Einstein Homogeneous Riemannian Fibrations

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To my godparents Amadeu and Conceição

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Abstract

This thesis is dedicated to the study of the existence of homogeneous Einstein metrics on the total space of homogeneous fibrations such that the fibers are totally geodesic manifolds. We obtain the Ricci curvature of an invariant metric with totally geodesic fibers and some necessary conditions for the existence of Einstein metrics with totally geodesic fibers in terms of Casimir operators. Some particular cases are studied, for instance, for normal base or fiber, symmetric fiber, Einstein base or fiber, for which the Einstein equations are manageable. We investigate the existence of such Einstein metrics for invariant bisymmetric fibrations of maximal rank, i.e., when both the base and the fiber are symmetric spaces and the base is an isotropy irreducible space of maximal rank. We find this way new Einstein metrics. For such spaces we describe explicitly the isotropy representation in terms subsets of roots and compute the eigenvalues of the Casimir operators of the fiber along the horizontal direction. Results for compact simply connected 4-symmetric spaces of maximal rank follow from this. Also, new invariant Einstein metrics are found on Kowalski n-symmetric spaces.

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Introduction

The principal topic of study in this thesis is the existence of Einstein invariant metrics on the total space of homogeneous Riemannian fibrations such that the fibers are totally geodesic submanifolds. In chapter 1 we introduce some main definitions and notation and deduce some essential formulas for the Ricci curvature of an invariant metric. Then we consider a fibration $G/L \to G/K$ such that G is a compact connected semisimple Lie group G, and $L \subsetneq K \subsetneq G$ are connected closed non-trivial subgroups of G. We assume that G/L has simple spectrum. On the total space G/L, we consider a G-invariant Riemannian metric such that the natural projection $G/L \to G/K$ is a Riemannian submersion and the fibers are totally geodesic submanifolds. We shall briefly call such a metric an *Einstein adapted metric*. We describe the Ricci curvature of any adapted metric in terms of Casimir operators and obtain two necessary conditions for existence of an Einstein adapted metric expressed only in terms of the Casimir operators of the horizontal and vertical directions. These will provide a tool to show that in many cases such a metric cannot exist without further study, i.e., only by studying eigenvalues of certain Casimir operators.

In chapter 2 we restrict the object of study to some special cases, where the Einstein equations are simpler. We consider the cases where the metric on the fiber or on the base is a multiple of the Killing form of G, in which cases are included those with isotropy irreducible fiber or base. The case when both these two conditions are satisfied gives rise to the study of the, throughout called, Einstein binormal metrics. The existence of Einstein binormal metrics translates into very simple algebraic conditions which shall allow us to find out new Einstein metrics. Also we obtain necessary conditions for existence of an Einstein adapted metric such that the metric on the base space or the metric on the fiber are also Einstein. Finally, we apply the results obtained so far to the case when the fiber is a symmetric space and N is isotropy irreducible.

Chapter 3 is devoted to bisymmetric fibrations of maximal rank, i.e., we consider a fibration $G/L \to G/K$, as in chapter 1, such that L is a subgroup of maximal rank, K/L is a symmetric space and G/L is an isotropy irreducible symmetric space. We introduce the notion of a bisymmetric triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ associated to a bisymmetric fibration. We obtain all the bisymmetric triples $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ in the case when $\mathfrak{g}$ is a simple Lie algebra and classify them into two different types, I and II. We classify all the Einstein adapted metrics when $\mathfrak{g}$ is an exceptional Lie algebra, for both Type I and II. When $\mathfrak{g}$ is a classical Lie algebra, we classify all the Einstein adapted metrics for Type I. For Type II in the classical case, we classify all Einstein binormal metrics and all Einstein adapted metrics whose restriction to the fiber is also Einstein; moreover, if one of these metrics exists we obtain all the other Einstein adapted metrics. Finally, we apply the results obtained to compact simply-connected irreducible 4-symmetric spaces. In appendix A we obtain all the necessary eigenvalues for the Einstein equations for each bisymmetric triple considered in this chapter.

In chapter 4 we study the existence of Einstein adapted metrics on the Kowalski n-symmetric spaces, i.e., we consider a fibration

$$\frac{\triangle^p G_0 \times \triangle^q G_0}{\triangle^n G_0} \to \frac{G_0^n}{\triangle^n G_0} \to \frac{G_0^p}{\triangle^p G_0} \times \frac{G_0^q}{\triangle^q G_0},$$

where G_0 is compact connected simple Lie group and $\triangle^m G_0$ is the diagonal subgroup in G_0^m, for $m = p, q, n$. It is known that $\frac{G_0^n}{\triangle^n G_0}$ is a standard Einstein manifold and we prove that, for $n > 4$, there exists another Einstein adapted metric, whereas, for $n = 4$, the standard metric is the only Einstein adapted metric.

CHAPTER 1

In Section 1 we introduce some essential definitions and notation. We deduce a formula for the Ricci curvature of an invariant metric on a reductive homogeneous space. In Section 2 we obtain the Ricci curvature of an invariant metric with totally geodesic fibers on the total space of a homogeneous fibration and some necessary conditions for that metric to be Einstein.

1.1 The Ricci Curvature of a Riemannian Homogeneous Space

A Riemannian metric g is said to be Einstein if its Ricci curvature satisfies an equation of the form $Ric = Eg$, for some constant E, the Einstein constant of g ([10]). This equation, commonly called the Einstein equation, is in general a very complicated system of partial differential equations of second order. Although so far no fully general results are known for existence of Einstein metrics, many results of existence and classification are known for many classes of spaces. Two examples of this are the Kähler-Einstein metrics ([49], [5], [40], [43]) and the Sasakian-Einstein metrics ([11]). Many results are known on homogeneous Einstein metrics. For Riemannian homogeneous spaces the Einstein equation translates into a system of algebraic equations, which is an easier problem than its general version. However, even for this class of spaces we are far from knowing full answers. Einstein normal homogeneous manifolds were classified by Wang and Ziller ([44]). Nowadays, it is known that every compact simply connected homogeneous manifold with dimension less or equal to 11 admits a homogeneous Einstein metric: any such manifold with dimension 2 or 3 has constant sectional curvature [10]; in dimension 4, the result was shown by Jensen ([18]), and by Alekseevsky, Dotti and Ferraris in dimension 5 ([4]); in dimension 6, the result is due to Nikonorov and Rodionov ([31]), and in dimension 7 it is due to Castellani, Romans and Warner ([14]). All the 7-dimensional homogeneous Einstein manifolds ([29]) were obtained by Nikonorov. These results were extended to dimension up to 11 by Böhm and Kerr ([12]). Many examples of homogeneous Einstein manifolds with dimension arbitrary big are known. Spheres and projective spaces are examples of this, where all homogeneous Einstein metrics were classified by Ziller ([51]). Also isotropy irreducible spaces ([47]), symmetric spaces ([16],[22]), flag manifolds, among many others, provide examples of Einstein manifolds with arbitrary big dimension. Einstein homogeneous fibrations have also been the object of study. We recall the work of Jensen on principal fibers bundles ([19]), where new invariant Einstein metrics are found on the total space of certain homogeneous fibrations, and the work of Wang and Ziller on principal torus bundles ([46]). Einstein homogeneous fibrations are the main focus of this thesis.

Let G be a Lie group and L a closed subgroup. We denote by $\mathfrak{g}$ and $\mathfrak{l}$ the Lie algebras of G and L, respectively. The homogeneous space $M = G/L$ is the space of all cosets $\{aL : a \in G\}$ endowed with the unique differentiable structure such that the canonical projection

$$\begin{array}{rccc} \pi : & G & \longrightarrow & M \\ & a & \longmapsto & aL \end{array} \tag{1.1}$$

is a submersion, i.e., $d\pi_a$ is onto for all $a \in G$, and with the natural transitive left action of G,

$$\begin{array}{rccc} \tau : & G \times M & \longrightarrow & M \\ & (b, aL) & \longmapsto & (ba)L \end{array} \tag{1.2}$$

Let $X \in \mathfrak{g}$ and let $exp\, tX$ be the one-parameter subgroup generated by X. For every $a \in G$,

$$d\pi_a(X) = \frac{d}{dt}(exp\, tX)aL\,|_{t=0} \tag{1.3}$$

and, in particular, for $o = \pi(e) = L$, this map yields an isomorphism

$$d\pi_e : \mathfrak{g}/\mathfrak{l} \cong T_oM. \tag{1.4}$$

For every $X \in \mathfrak{g}$ we define a G-invariant vector field on M by

$$X^*_{aL} = d\pi_a(X) = \frac{d}{dt}(exp\, tX)gL\,|_{t=0}\ . \tag{1.5}$$

The homogeneous space M is called *reductive* if there exists a direct complement $\mathfrak{m}$ of $\mathfrak{l}$ in $\mathfrak{g}$ which is $Ad\, L$-invariant, i.e.,

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m} \text{ and } Ad\, L(\mathfrak{m}) \subset \mathfrak{m}. \tag{1.6}$$

The inclusion $Ad\, L(\mathfrak{m}) \subset \mathfrak{m}$ implies

$$[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m} \tag{1.7}$$

and the converse holds if L is connected. The homogeneous space M is reductive if L is compact. Throughout, we suppose that L is compact and we denote by $\mathfrak{m}$ a reductive complement of $\mathfrak{l}$ on $\mathfrak{g}$. If M is reductive we have an isomorphism

$$\mathfrak{m} \cong T_oM$$

and the tangent space T_oM is identified with $\mathfrak{m}$ and consequently the vector field X^* on M is identified with $X \in \mathfrak{m}$. Under this identification we shall simply write X for X^*_o. Furthermore, the isotropy representation of M,

$$Ad^M : L \to GL(T_oM),$$

is equivalent to the adjoint representation of L on $\mathfrak{m}$. Consequently, there is a one-to-one correspondence between G-invariant objects on M and $Ad\,L$-invariant objects on $\mathfrak{m}$. In particular, G-invariant metrics on M correspond to $Ad\,L$-invariant scalar products on $\mathfrak{m}$. More precisely, a metric g on M is said to be G-invariant if, for every $a \in G$,

$$\tau_a^* g = g$$

and the correspondence between G-invariant metrics on M and $Ad\,L$-scalar products on $\mathfrak{m}$ is given

$$g_a(X^*_a, Y^*_a) =< X, Y >, \text{ for all } a \in G. \tag{1.8}$$

Let $Kill$ be the Killing form of $\mathfrak{g}$. We recall that $Kill$ is the bilinear form on $\mathfrak{g}$ defined by

$$Kill(X, Y) = tr(ad_X ad_Y),\ X, Y \in \mathfrak{g} \tag{1.9}$$

where, for each $X \in \mathfrak{g}$, ad_X denotes the adjoint map

$$\begin{array}{ccc} \mathfrak{g} & \to & \mathfrak{g} \\ Y & \mapsto & [X, Y] \end{array}. \tag{1.10}$$

The Killing form of $\mathfrak{g}$ is an $Ad\,G$-invariant bilinear form and it is non-degenerate if $\mathfrak{g}$ is semisimple. Moreover, if G is a compact connected semisimple Lie group, $Kill$ is negative definite. In this case, by (1.8), the negative of the Killing form induces a G-invariant metric on M, the *standard Riemannian metric* on M.

With respect to the decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$, we write

$$ad_X = \begin{pmatrix} 0 & C_X \\ B_X & P_X \end{pmatrix}, \text{ for every } X \in \mathfrak{m}. \tag{1.11}$$

Hence, for $Y \in \mathfrak{m}$,

$$(ad_X Y)_{\mathfrak{m}} = P_X Y \text{ and } (ad_X^2 Y)_{\mathfrak{m}} = (B_X C_X + P_X^2)Y, \tag{1.12}$$

where the subscript $\mathfrak{m}$ denotes the projection onto $\mathfrak{m}$.

Let g_M be a G-invariant metric on M. As was explained above, there is a one-to-one correspondence between G-invariant metrics on M and $Ad\,L$-invariant scalar products on $\mathfrak{m}$. So let $<,>$ be the $Ad\,L$-invariant scalar product on $\mathfrak{m}$ corresponding to g_M.

For every $X \in \mathfrak{m}$, let T_X be the endomorphism of $\mathfrak{m}$ defined by

$$< T_X Y, Z >=< X, P_Y Z >, \text{ for every } Y, Z \in \mathfrak{m}. \tag{1.13}$$

The *Nomizu operator* of the scalar product $<,>$ (cf. [28], [25]) is

$$L_X \in End(\mathfrak{m}),\ X \in \mathfrak{g},$$

defined by

$$L_X Y = -\nabla_Y X^*,\ Y \in \mathfrak{m} \tag{1.14}$$

where ∇ is the Riemannian connection of g_M. We have

$$L_X Y = \frac{1}{2} P_X Y + U(X, Y), \tag{1.15}$$

where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is the operator

$$U(X, Y) = -\frac{1}{2}(T_X Y + T_Y X),\ X, Y \in \mathfrak{m}. \tag{1.16}$$

The metric g_M is called *naturally reductive* if $U = 0$. The curvature tensor, the sectional curvature and the Ricci curvature of g_M are G-invariant tensors and thus they are determined by the following identities ([28]), which represent their values at the point o. By using L the curvature tensor of g_M at o can be written as

$$R(X, Y) = [L_X, L_Y]_{\mathfrak{m}} - L_{[X,Y]_{\mathfrak{m}}} - ad_{[X,Y]_{\mathfrak{l}}}, \tag{1.17}$$

for every X, $Y \in \mathfrak{m}$. The sectional curvature K of g_M is defined by

$$K(Z, X) =< R(Z, X)X, Z >, \tag{1.18}$$

for every $X, Z \in \mathfrak{m}$ orthonormal with respect to $<,>$. The Ricci curvature of g_M is determined by

$$Ric(X,X) = \sum_i K(Z_i, X),\ X \in \mathfrak{m} \tag{1.19}$$

where $(Z_i)_i$ is an orthonormal basis of $\mathfrak{m}$ with respect to $<,>$. The metric g_M is said to be an *Einstein metric* if

$$Ric = E g_M, \tag{1.20}$$

for some constant E called the *Einstein constant* of g_M. Equipped with a G-invariant Einstein metric, M is called an Einstein homogeneous manifold.

Below we show some elementary properties of the Nomizu operator:

Lemma 1.1. *(i) L_X is skew-symmetric with respect to $<,>$, i.e.,*

$$< L_X Y, Z > + < Y, L_X Z >= 0,\ X, Y, \in \mathfrak{m}; \tag{1.21}$$

(ii) for every $X, Y \in \mathfrak{m}$,

$$L_X Y - L_Y X = [X, Y]_{\mathfrak{m}} = P_X Y. \tag{1.22}$$

Proof: From identities (1.13), (1.15) and (1.16) we deduce

$$\begin{aligned} 2 < L_X Y, Z >= &\ < P_X Y, Z > - < T_X Y, Z > - < T_Y X, Z > \\ = &\ < T_Z X, Y > + < T_X Z, Y > - < P_X Z, Y >= \\ = &\ -2 < Y, L_X Z > \end{aligned}$$

and this shows the skew-symmetry of L_X. To show (ii) we just observe that $U(X,Y) = U(Y,X)$ and $P_X Y = -P_Y X$. Equivalently, this assertion just means that the Levi-Civita connection is torsion free since $\nabla_{X^*} Y^* = L_X Y$.
□
For every $X, Y \in \mathfrak{m}$, we define the operator

$$R_X Y = L_Y X \tag{1.23}$$

and the vector

$$V_X = L_X X. \tag{1.24}$$

Lemma 1.2. *For every $X \in \mathfrak{m}$,*

$$R_X = -\frac{1}{2}(P_X + P_X^* + T_X) \text{ and } R_X^* = -\frac{1}{2}(P_X + P_X^* - T_X).$$

Proof: Let X, Y, $Z \in \mathfrak{m}$.

$$\begin{aligned} 2 < R_X Y, Z >= & \ 2 < L_Y X, Z > \\ = & \ < [Y, X]_{\mathfrak{m}}, Z > + < Y, [Z, X]_{\mathfrak{m}}, > + < X, [Y, Z]_{\mathfrak{m}} > \\ = & \ - < P_X Y, Z > - < P_X^* Y, Z > - < T_X Y, Z > . \end{aligned}$$

Thus we obtain the required expression for R_X. The formula for R_X^* follows from the fact that $P_X + P_X^*$ is symmetric and T_X is skew symmetric with respect to $<,>$.

□

Lemma 1.3. *The sectional curvature of g_M is*

$$K(Z, X) =< \big(R_X^* R_X - P_X^2 - P_X^* P_X - B_X C_X + P_{V_X}\big) Z, Z >,$$

for every Z, $X \in \mathfrak{m}$ orthonormal with respect to $<,>$.

Proof: Let Z, $X \in \mathfrak{m}$. The proof is a straightforward calculation by using the identities (1.17), (1.21) and (1.22):

$$\begin{aligned} K(Z, X) = & \ < R(Z, X) X, Z > \\ = & \ < [L_Z, L_X]_{\mathfrak{m}} X, Z > - < L_{[Z,X]_{\mathfrak{m}}} X, Z > - < ad_{[Z,X]_{\mathfrak{l}}} X, Z > \\ = & \ < L_Z L_X X, Z > - < L_X L_Z X, Z > - < L_X [Z, X]_{\mathfrak{m}}, Z > - \\ & - < [[Z, X]_{\mathfrak{m}}, X]_{\mathfrak{m}}, Z > - < [[Z, X]_{\mathfrak{l}} X]_{\mathfrak{m}}, Z > \\ = & \ - < L_X X, L_Z Z, > + < L_Z X, L_X Z > + < [Z, X]_{\mathfrak{m}}, L_X Z > - \\ & - < [[Z, X], X]_{\mathfrak{m}}, Z > \\ = & \ - < L_X X, L_Z Z, > + < L_Z X, L_Z X > + < L_Z X, [X, Z]_{\mathfrak{m}} > + \\ & + < [Z, X]_{\mathfrak{m}}, L_Z X > + < [Z, X]_{\mathfrak{m}}, [X, Z]_{\mathfrak{m}} > - \\ & - < [X, [X, Z]]_{\mathfrak{m}}, Z > \\ = & \ - < L_X X, L_Z Z, > + < L_Z X, L_Z X > - < [Z, X]_{\mathfrak{m}}, [Z, X]_{\mathfrak{m}} > - \\ & - < (P_X^2 Z)_{\mathfrak{m}}, Z > \end{aligned}$$

$$= \; < L_Z X, L_Z X > - < [X,Z]_{\mathfrak{m}}, [X,Z]_{\mathfrak{m}} > - < (P_X^2 Z)_{\mathfrak{m}}, Z >$$

$$- < L_Z V_X, Z >$$

$$= \; < R_X Z, R_X Z > - < P_X Z, P_X Z > - < (P_X^2 + B_X C_X) Z, Z >$$

$$- < L_{V_X} Z + [Z, V_X]_{\mathfrak{m}}, Z >$$

$$= \; < (R_X^* R_X - P_X^* P_X - B_X C_X - P_X^2) Z, Z > + < (P_{V_X} - L_{V_X}) Z, Z > .$$

Since L_X is skew-symmetric with respect to $<,>$, we have $< L_{V_X} Z, Z >= 0$, for every $Z \in \mathfrak{m}$, and we obtain

$$K(Z,X) =< \big(R_X^* R_X - P_X^* P_X - B_X C_X - P_X^2 + P_{V_X}\big) Z, Z >,$$

as required.
$\square$

Theorem 1.1. *Let $X, Y \in \mathfrak{m}$. Then*

$$Ric(X,Y) = -\frac{1}{2} tr(2R_X R_Y + B_X C_Y + B_Y C_X - 2P_{U(X,Y)}).$$

In particular,

$$Ric(X,X) = -tr(R_X^2 + B_X C_X - P_{V_X}).$$

Proof: We first compute $Ric(X,X)$ and then obtain $Ric(X,Y)$ by polarization.

Since T_X is a skew-symmetric operator and $P_X + P_X^*$ is symmetric,

$$tr\big((P_X + P_X^*) T_X\big) = -tr\big(T_X (P_X + P_X^*)\big) = tr\big((P_X + P_X^*) T_X\big)$$

and thus all the terms vanish. Therefore, by using Lemma 1.2 we obtain

$$tr(R_X^* R_X) = \frac{1}{4}\big((P_X + P_X^*)^2 - T_X^2\big) = \frac{1}{4} tr(2P_X^2 + 2P_X^* P_X - T_X^2)$$

and

$$tr(R_X^2) = \frac{1}{4}\big((P_X + P_X^*)^2 + T_X^2\big) = \frac{1}{4} tr(2P_X^2 + 2P_X^* P_X + T_X^2).$$

Hence, $tr(R_X^* R_X - P_X^2 - P_X^* P_X) = -tr(R_X^2)$.

Let us suppose that $< X, X >= 1$ and let $\{e_i\}_i$ be an orthonormal basis for $\mathfrak{m}$ with respect to $<,>$ such that $X = e_1$. Hence, we can apply Lemma 1.3 to obtain the following:

$$\begin{aligned} Ric(X,X) = & \sum_i K(e_i,X) \\ = & < \big(R_X^* R_X - P_X^* P_X - B_X C_X - P_X^2 + P_{V_X}\big) e_i, e_i > \\ = & tr\big(R_X^* R_X - P_X^2 - P_X^* P_X - B_X C_X + P_{V_X}\big) \\ = & -tr(R_X^2 + B_X C_X - P_{V_X}). \end{aligned}$$

Since both Ric and the map $X \mapsto tr(R_X^2 + B_X C_X - P_{V_X})$ are bilinear maps, the identity above holds even if X is not unit. Hence, for every $X \in \mathfrak{m}$,

$$Ric(X,X) = -tr(R_X^2 + B_X C_X - P_{V_X}).$$

Now we compute $Ric(X,Y)$. Since Ric is a symmetric bilinear operator we have

$$2Ric(X,Y) = Ric(X+Y, X+Y) - Ric(X,X) - Ric(Y,Y).$$

By using the expression above for $Ric(X,X)$ we get

$$\begin{aligned} 2Ric(X,Y) = & tr(-R_{X+Y}^2 + R_X^2 + R_Y^2) + \\ + & tr(-B_{X+Y} C_{X+Y} + B_X C_X + B_Y C_Y) + \\ + & tr(P_{V_{X+Y}} - P_{V_X} - P_{V_Y}). \end{aligned}$$

By bilinearity of L and property (1.22) we obtain

$$\begin{aligned} V_{X+Y} = & L_{X+Y}(X+Y) \\ = & L_X X + L_Y Y + L_X Y + L_Y X \\ = & V_X + V_Y + 2L_X Y - [X,Y]_{\mathfrak{m}} \\ = & V_X + V_Y + 2L_X Y + P_X Y. \end{aligned}$$

Therefore, $P_{V_{X+Y}} - P_{V_X} - P_{V_Y} = P_{2L_X Y - P_X Y}$.

Also the identity $B_{X+Y} C_{X+Y} = B_X C_X + B_Y C_Y + B_X C_Y + B_Y C_X$ implies that

$$-B_{X+Y} C_{X+Y} + B_X C_X + B_Y C_Y = -B_X C_Y - B_Y C_X.$$

Moreover, $R_{X+Y}^2 = R_X^2 + R_Y^2 + R_X R_Y + R_Y R_X$ and thus

$$tr(-R_{X+Y}^2 + R_X^2 + R_Y^2) = -tr(R_X R_Y + R_Y R_X) = -tr(2R_X R_Y).$$

Therefore, we obtain

$$\begin{aligned} 2Ric(X,Y) = & -tr(2R_XR_Y + B_XC_Y + B_YC_X) + tr(P_{2L_XY - P_XY}) \\ = & -tr(2R_XR_Y + B_XC_Y + B_YC_X) + tr(P_{2U(X,Y)}). \end{aligned}$$

□

Corollary 1.1. *Let $X, Y \in \mathfrak{m}$.*

$$Ric(X,Y) = -\frac{1}{4}tr(2P_X^*P_Y + T_XT_Y) - \frac{1}{2}Kill(X,Y) + tr(P_{U(X,Y)}).$$

Proof: By using Theorem 1.1 and Lemma 1.2, we write Ric as follows:

$$Ric(X,Y) = -\frac{1}{2}\left(\frac{1}{2}(P_X + P_X^* + T_X)(P_Y + P_Y^* + T_Y) + B_XC_Y + B_YC_X - 2P_{U(X,Y)}\right). \tag{1.25}$$

Since $P_X + P_X^*$ and $P_Y + P_Y^*$ are symmetric linear maps and T_X and T_Y are skew-symmetric, we have

$$tr((P_X + P_X^*)T_Y) = tr(T_X(P_Y + P_Y^*)) = 0. \tag{1.26}$$

Moreover,

$$tr(P_XP_Y) = tr(P_X^*P_Y^*)) \text{ and } tr(P_X^*P_Y) = tr(P_XP_Y^*)). \tag{1.27}$$

We can use (1.11) to write $Kill$ as follows:

$$Kill(X,Y) = tr(ad_Xad_Y) = tr(C_XB_Y + B_XC_Y + P_XP_Y). \tag{1.28}$$

Finally, by using (1.26), (1.27) and (1.28) we simplify (1.25) to obtain the expression stated for Ric.

□

Definition 1.1. *A symmetric bilinear map β on $\mathfrak{m}$ is said to be associative if $\beta([u,v]_{\mathfrak{m}}, w) = \beta(u, [v,w]_{\mathfrak{m}})$, for every $u, v, w \in \mathfrak{m}$.*

Remark 1.1. *If there exists on $\mathfrak{m}$ an associative symmetric bilinear form β such that β is non-degenerate, then $tr(P_{U(X,Y)}) = 0$, for all $X, Y \in \mathfrak{m}$. Indeed, if such a bilinear form exists, $tr\, P_a = 0$, for every $a \in \mathfrak{m}$. Let $\{w_i\}_i$ and $\{w_i'\}_i$ be bases of $\mathfrak{m}$ dual with respect to β, i.e., $\beta(w_i, w_j') = \delta_{ij}$. Then, for every $a \in \mathfrak{m}$,*

$$\begin{aligned} \beta(P_aw_i, w_i') = & \beta([a,w_i]_{\mathfrak{m}}, w_i') \\ = & -\beta(w_i, [a, w_i']_{\mathfrak{m}}) \\ = & -\beta(P_aw_i', w_i). \end{aligned}$$

Hence, $tr(P_a) = 0$. Also, if the metric g_M on M is naturally reductive, then $P_{U(X,Y)} = 0$, for all $X, Y \in \mathfrak{m}$, since, in this case, U is identically zero.
$\diamond$

Definition 1.2. *Let U, V be $Ad\,L$-invariant vector subspaces of $\mathfrak{g}$. We define a bilinear map $Q^{UV} : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ by*

$$Q^{UV}_{XY} = tr([X, [Y, \cdot]_V]_U),\ X, Y \in \mathfrak{m},$$

where the subscripts U and V denote the projections onto U and V, respectively.

Definition 1.3. *Let U be an $Ad\,L$-invariant vector subspace of $\mathfrak{g}$ such that the restriction of Kill to U is non-degenerate. The Casimir operator of U with respect to the Killing form of $\mathfrak{g}$ is the operator*

$$C_U = \sum_i ad_{u_i} ad_{u'_i},$$

where $\{u_i\}_i$ and $\{u'_i\}_i$ are bases of U which are dual with respect to Kill, i.e., $Kill(u_i, u'_j) = \delta_{ij}$.

The Casimir operator C_U is an $Ad\,L$-invariant linear map and thus it is scalar on any irreducible $Ad\,L$-module. In particular, if $\mathfrak{g}$ is simple, then $C_{\mathfrak{g}} = Id_{\mathfrak{g}}$.

Lemma 1.4. *Suppose that $\mathfrak{g}$ is semisimple. Let U, V be $Ad\,L$-invariant vector subspaces of $\mathfrak{g}$ such that the restrictions of the Killing form to U and V are both non degenerate.*
(i) Q^{UV} is an $Ad\,L$-invariant symmetric bilinear map. Hence, if $W \subset \mathfrak{g}$ is any irreducible $Ad\,L$-submodule, then $Q^{UV}\mid_{W\times W}$ is a multiple of $Kill\mid_{W\times W}$.
(ii) $Q^{UV} = Q^{VU}$.

Proof: Since $\mathfrak{g}$ is a semisimple Lie algebra its Killing form is non-degenerate. As in addition $Kill\mid_{U\times U}$ and $Kill\mid_{V\times V}$ are non-degenerate, we may consider the orthogonal complements $U^\perp$ and $V^\perp$ of U and V, respectively, in $\mathfrak{g}$ with respect to $Kill$. It follows that

$$Kill\mid_{U\times\mathfrak{g}} = Kill(\cdot, p_U\cdot)\mid_{U\times\mathfrak{g}}$$

and

$$Kill\mid_{V\times\mathfrak{g}} = Kill(\cdot, p_V\cdot)\mid_{V\times\mathfrak{g}},$$

where p_U and p_V are the projections onto U and V, respectively.

Also, we may consider bases $\{w_i\}_i$ and $\{w_i'\}_i$ of U which are dual with respect to $Kill$. By using these facts we have the following:

(i) Let $X, Y \in \mathfrak{m}$ and $g \in L$.

$$\begin{aligned} Kill([X,[Y,w_i]_V]_U, w_i') = &\ Kill([X,[Y,w_i]_V], w_i') \\ = &\ -Kill([Y,w_i]_V, [X,w_i']) \\ = &\ -Kill([Y,w_i], [X,w_i']_V) \\ = &\ Kill(w_i, [Y,[X,w_i']_V]) \\ = &\ Kill(w_i, [Y,[X,w_i']_V]_U). \end{aligned}$$

Therefore, $tr([X,[Y,\cdot]_V]_U) = tr([Y,[X,\cdot]_V]_U)$ and thus $Q^{UV}_{XY} = Q^{UV}_{YX}$. So Q^{UV} is symmetric.

To show the $Ad\,L$-invariance of Q^{UV} we note that since V and $V^\perp$ are $Ad\,L$-invariant subspaces and $\mathfrak{g} = V \oplus V^\perp$, the projections on V and $V^\perp$ are also $Ad\,L$-invariant linear maps.

$$\begin{aligned} Kill([Ad_gX,[Ad_gY,w_i]_V]_U, w_i') = &\ Kill([Ad_gX,[Ad_gY,w_i]_V], w_i') \\ = &\ Kill(Ad_{g^{-1}}[Ad_gX,[Ad_gY,w_i]_V], Ad_{g^{-1}}w_i') \\ = &\ Kill([X, Ad_{g^{-1}}[Ad_gY,w_i]_V], Ad_{g^{-1}}w_i') \\ = &\ Kill([X,[Y, Ad_{g^{-1}}w_i]_V], Ad_{g^{-1}}w_i') \\ = &\ Kill([X,[Y, Ad_{g^{-1}}w_i]_V]_U, Ad_{g^{-1}}w_i'). \end{aligned}$$

If $\{w_i\}_i$ and $\{w_i'\}_i$ are dual bases of U with respect to $Kill$, then $\{Ad_{g^{-1}}w_i\}_i$ and $\{Ad_{g^{-1}}w_i'\}_i$ are still dual bases as well since the Killing form is invariant under inner automorphisms. So by the above we conclude that

$$tr([Ad_gX,[Ad_gY,\cdot]_V]_U) = tr([X,[Y,\cdot]_V]_U)$$

and thus Q^{UV} is $Ad\,L$-invariant.

(ii) For $Z \in \mathfrak{m}$, let $A_Z = (ad_Z\,|_U)_V$ and $B_Z = (ad_Z\,|_V)_U$. We have

$$Q^{VU}_{XY} = tr(A_XB_Y) = tr(B_YA_X) = Q^{UV}_{YX}.$$

Hence, by symmetry of Q^{UV}, we conclude that $Q^{VU}_{XY} = Q^{UV}_{YX} = Q^{UV}_{XY}$, for every $X, Y \in \mathfrak{m}$. Therefore, $Q^{UV} = Q^{VU}$.

□

Lemma 1.5. *Suppose that $\mathfrak{g}$ is semisimple. Let U, V be $Ad\,L$-invariant vector subspaces of $\mathfrak{g}$ such that the restrictions of the Killing form to U and V are both non-degenerate. For every $X, Y \in \mathfrak{m}$,*
(i) if $ad_X U \subset V$ or $ad_Y U \subset V$, then $Q^{UV}_{XY} = Kill(C_U X, Y) = Kill(X, C_U Y)$;
(ii) if $ad_X V \perp U$ or $ad_Y V \perp U$, then $Q^{UV}_{XY} = 0$;
(iv) if $ad_X ad_Y U \perp U$ or $ad_Y ad_X U \perp U$, then $Q^{UV}_{XY} = 0$.

Proof: (i) Let $C_U = \sum_i ad_{w_i} ad_{w'_i}$ be the Casimir operator of U. Since $Q^{UV} = Q^{VU}$ it suffices to suppose that $ad_Y U \subset V$. If $ad_Y U \subset V$, then

$$Q^{UV}_{XY} = tr([X, [Y, \cdot]]_U) = tr(ad_X ad_Y \mid_U).$$

Since

$$\begin{aligned} Kill([X,[Y,w_i]]_U, w'_i) = & \ Kill([X,[Y,w_i]], w'_i) \\ = & \ -Kill([Y,w_i],[X,w'_i]) \\ = & \ Kill(Y,[w_i,[w'_i,X]]), \end{aligned}$$

we have $Q^{UV}_{XY} = \sum_i Kill(Y, [w_i, [w'_i, X]]) = Kill(Y, C_U X)$. By symmetry of Q^{UV} we also get $Q^{UV}_{XY} = Kill(X, C_U Y)$.

(ii) If $ad_X V \perp U$, then, for every $w, w' \in U$, $Kill([X, [Y, w]_V], w') = 0$ and thus $Q^{UV}_{XY} = 0$, for every $Y \in \mathfrak{m}$. By symmetry, the same conclusion holds if $ad_Y V \perp U$.

(iii) If $ad_X ad_Y U \perp U$, then, for every $w, w' \in U$, $Kill([X, [Y, w]], w') = 0$ and thus $Kill([X, [Y, w]_V]_U, w') = 0$. Hence $Q^{UV}_{XY} = 0$. If $ad_Y ad_X U \perp U$, then $Q^{UV}_{XY} = 0$ by symmetry.
□

Remark 1.2. *In Lemmas 1.4 and 1.5 the condition that $\mathfrak{g}$ is semisimple may be replaced by requiring that there is on $\mathfrak{g}$ a non-degenerate $Ad\,L$-invariant symmetric bilinear form β, since in the proofs above the Killing form may be replaced by any such form β. In this case, the orthogonality conditions in Lemma 1.5 should be understood as conditions with respect to β.*

Theorem 1.2. *Let β be an associative $Ad\,L$-invariant non-degenerate symmetric bilinear form on $\mathfrak{m}$. Let $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_m$ be a decomposition of $\mathfrak{m}$ into $Ad\,L$-invariant subspaces such that $\beta \mid_{\mathfrak{m}_j \times \mathfrak{m}_i} = 0$, if $i \neq j$. Let g_M be the G-invariant pseudo-Riemannian metric on G/L induced by the scalar product of the form*

$$<,> = \oplus_{j=1}^m \nu_j \beta \mid_{\mathfrak{m}_j \times \mathfrak{m}_j}, \tag{1.29}$$

for $\nu_j > 0$, for every $j = 1, \ldots, m$. For every $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_b$, the Ricci curvature of g_M is given as follows:

$$Ric(X,Y) = \frac{1}{2}\sum_{j,k=1}^{m}\left(\frac{\nu_k}{\nu_j} - \frac{\nu_a\nu_b}{2\nu_k\nu_j}\right)Q_{XY}^{\mathfrak{m}_j\mathfrak{m}_k} - \frac{1}{2}Kill(X,Y).$$

Proof: First we observe that the non-degeneracy of β and the condition of pairwise orthogonality of the $\mathfrak{m}_j$'s imply that $\beta\mid_{\mathfrak{m}_j\times\mathfrak{m}_j}$ is in fact non-degenerate. Let $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_b$. By Corollary 1.1 we have

$$Ric(X,Y) = -\frac{1}{4}tr(2P_X^*P_Y + T_XT_Y) - \frac{1}{2}Kill(X,Y) + tr(P_{U(X,Y)}).$$

According to Remark 1.1, we have $tr(P_{U(X,Y)}) = 0$.

Let $j = 1, \ldots, m$ and let $\{w_i\}_i$ and $\{w_i'\}_i$ be dual bases for $\mathfrak{m}_j$ with respect to β. We note that such bases exist as $\beta\mid_{\mathfrak{m}_j\times\mathfrak{m}_j}$ is non-degenerate.

$$\begin{aligned}
< T_XT_Yw_i, w_i' > &= < X, [T_Yw_i, w_i']_{\mathfrak{m}} > \\
&= \nu_a\beta(X, [T_Yw_i, w_i']) \\
&= -\nu_a\beta(T_Yw_i, [X, w_i']) \\
&= -\nu_a \textstyle\sum_{k=1}^m \beta(T_Yw_i, [X, w_i']_{\mathfrak{m}_k}) \\
&= -\nu_a \textstyle\sum_{k=1}^m \frac{1}{\nu_k} < T_Yw_i, [X, w_i']_{\mathfrak{m}_k} > \\
&= -\nu_a \textstyle\sum_{k=1}^m \frac{1}{\nu_k} < Y, [w_i, [X, w_i']_{\mathfrak{m}_k}]_{\mathfrak{m}} > \\
&= -\nu_a\nu_b \textstyle\sum_{k=1}^m \frac{1}{\nu_k}\beta(Y, [w_i, [X, w_i']_{\mathfrak{m}_k}]) \\
&= -\nu_a\nu_b \textstyle\sum_{k=1}^m \frac{1}{\nu_k}\beta([Y, w_i], [X, w_i']_{\mathfrak{m}_k}) \\
&= -\nu_a\nu_b \textstyle\sum_{k=1}^m \frac{1}{\nu_k}\beta([Y, w_i]_{\mathfrak{m}_k}, [X, w_i']) \\
&= \nu_a\nu_b \textstyle\sum_{k=1}^m \frac{1}{\nu_k}\beta(w_i', [X, [Y, w_i]_{\mathfrak{m}_k}]) \\
&= \nu_a\nu_b \textstyle\sum_{k=1}^m \frac{1}{\nu_k}\beta(w_i', [X, [Y, w_i]_{\mathfrak{m}_k}]_{\mathfrak{m}_j}) \\
&= \nu_a\nu_b \textstyle\sum_{k=1}^m \frac{1}{\nu_k\nu_j} < w_i', [X, [Y, w_i]_{\mathfrak{m}_k}]_{\mathfrak{m}_j} > .
\end{aligned}$$

Hence,

$$tr(T_X T_Y\mid_{\mathfrak{m}_j}) = \nu_a\nu_b \sum_{k=1}^m \frac{1}{\nu_k\nu_j} tr([X,[Y,\cdot]_{\mathfrak{m}_k}]_{\mathfrak{m}_j}) = \nu_a\nu_b \sum_{k=1}^m \frac{1}{\nu_k\nu_j} Q_{XY}^{\mathfrak{m}_j\mathfrak{m}_k}$$

and thus

$$tr(T_X T_Y) = \nu_a\nu_b \sum_{j,k=1}^m \frac{1}{\nu_k\nu_j} Q_{XY}^{\mathfrak{m}_j\mathfrak{m}_k}.$$

$$\begin{aligned}
< P_X^* P_Y w_i, w_i' > &= < P_Y w_i, P_X w_i' > \\
&= \textstyle\sum_{k=1}^m < [Y,w_i]_{\mathfrak{m}_k}, [X,w_i']_{\mathfrak{m}_k} > \\
&= \textstyle\sum_{k=1}^m \nu_k \beta([Y,w_i]_{\mathfrak{m}_k}, [X,w_i']) \\
&= -\textstyle\sum_{k=1}^m \nu_k \beta(w_i', [X,[Y,w_i]_{\mathfrak{m}_k}]) \\
&= -\textstyle\sum_{k=1}^m \nu_k \beta(w_i', [X,[Y,w_i]_{\mathfrak{m}_k}]_{\mathfrak{m}_j}) \\
&= -\textstyle\sum_{k=1}^m \dfrac{\nu_k}{\nu_j} < w_i', [X,[Y,w_i]_{\mathfrak{m}_k}]_{\mathfrak{m}_j} > .
\end{aligned}$$

Then

$$tr(P_X^* P_Y\mid_{\mathfrak{m}_j}) = -\sum_{k=1}^m \frac{\nu_k}{\nu_j} tr([X,[Y,\cdot]_{\mathfrak{m}_k}]_{\mathfrak{m}_j}) = -\sum_{k=1}^m \frac{\nu_k}{\nu_j} Q_{XY}^{\mathfrak{m}_j\mathfrak{m}_k}$$

and thus we get

$$tr(P_X^* P_Y) = -\sum_{j,k=1}^m \frac{\nu_k}{\nu_j} Q_{XY}^{\mathfrak{m}_j\mathfrak{m}_k}.$$

By using Corollary 1.1 we finally obtain the required expression for $Ric(X,Y)$.
□

We recall that a metric g_M is said to be *normal* if it is a multiple of an associative $Ad\, L$-invariant non-degenerate symmetric bilinear form on $\mathfrak{m}$.

Corollary 1.2. *If g_M is a normal metric, then $Ric(\mathfrak{m}_i,\mathfrak{m}_j) = 0$, for all $i \neq j$. For every $X \in \mathfrak{m}_j$,*

$$Ric(X,X) = -\frac{1}{4} Kill(X,X) - \frac{1}{2} Kill(C_{\mathfrak{l}} X, X),$$

where $C_{\mathfrak{l}}$ is the Casimir operator of $\mathfrak{l}$ with respect to the Killing form. Furthermore, if $\mathfrak{m}_j$ is irreducible, then

$$Ric\mid_{\mathfrak{m}_j\times\mathfrak{m}_j} = -\frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{l},j}\right) Kill\mid_{\mathfrak{m}_j\times\mathfrak{m}_j},$$

where $c_{\mathfrak{l},j}$ is the eigenvalue of $C_{\mathfrak{l}}$ on $\mathfrak{m}_j$.

Proof: Let $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_b$. If g_M is a normal metric, then there exists an $Ad\,L$-invariant non-degenerate symmetric bilinear form β on $\mathfrak{m}$ which induces g_M. Hence, in Theorem 1.2 we can take $\nu_1 = \ldots = \nu_m = 1$ and obtain the following:

$$\begin{aligned} Ric(X,Y) = &\ \tfrac{1}{4}\textstyle\sum_{j,k=1}^{m} Q_{XY}^{\mathfrak{m}_j\mathfrak{m}_k} - \tfrac{1}{2}Kill(X,Y) \\ = &\ \tfrac{1}{4}Q_{XY}^{\mathfrak{m}\mathfrak{m}} - \tfrac{1}{2}Kill(X,Y) \\ = &\ \tfrac{1}{4}Q_{XY}^{\mathfrak{m}\mathfrak{g}} - \tfrac{1}{4}Q_{XY}^{\mathfrak{m}\mathfrak{l}} - \tfrac{1}{2}Kill(X,Y) \\ = &\ \tfrac{1}{4}Q_{XY}^{\mathfrak{m}\mathfrak{g}} - \tfrac{1}{4}Q_{XY}^{\mathfrak{l}\mathfrak{m}} - \tfrac{1}{2}Kill(X,Y) \\ = &\ \tfrac{1}{4}Kill(C_{\mathfrak{m}}X,Y) - \tfrac{1}{4}Kill(C_{\mathfrak{l}}X,Y) - \tfrac{1}{2}Kill(X,Y) \\ = &\ -\tfrac{1}{4}Kill(X,Y) - \tfrac{1}{2}Kill(C_{\mathfrak{l}}X,Y). \end{aligned}$$

Since $C_{\mathfrak{l}}(\mathfrak{m}_a) \subset \mathfrak{m}_a$, it is clear that $Ric(X,Y) = 0$ if $a \neq b$ and Ric is well determined by elements $Ric(X,X)$ with $X \in \mathfrak{m}_a$. If $\mathfrak{m}_j$ is irreducible, then $C_{\mathfrak{l}}$ is scalar on $\mathfrak{m}_j$ and we obtain the identity given for Ric.

□

The formula above for the Ricci curvature of a normal metric was first found by M.Y. Wang and W. Ziller in [44]. From Corollary 1.2, it is clear that a necessary and sufficient condition for a normal metric to be Einstein is that the Casimir operator of $\mathfrak{l}$ is scalar on the isotropy space $\mathfrak{m}$. For instance, this condition holds if $\mathfrak{m}$ is irreducible. Simply connected non-strongly isotropy irreducible homogeneous spaces which admit a normal Einstein metric were classified by M.Y. Wang and W. Ziller in [44], when G is a compact connected simple group. Also, more generally, simply connected compact standard homogeneous manifolds were studied by E.D. Rodionov in [39].

We obtain a similar formula to that of Corollary 1.2, in the case when the submodules $\mathfrak{m}_1, \ldots, \mathfrak{m}_m$ pairwise commute.

Corollary 1.3. *If $\mathfrak{m}_1, \ldots, \mathfrak{m}_m$ pairwise commute, i.e., $[\mathfrak{m}_a, \mathfrak{m}_b] = 0$, for all $a \neq b$, then $Ric(\mathfrak{m}_a, \mathfrak{m}_b) = 0$, for all $a \neq b$. For every $X \in \mathfrak{m}_a$,*

$$Ric(X,X) = -\frac{1}{4}Kill(X,X) - \frac{1}{2}Kill(C_{\mathfrak{l}}X,X),$$

where $C_{\mathfrak{l}}$ is the Casimir operator of $\mathfrak{l}$ with respect to the Killing form. Furthermore, if $\mathfrak{m}_j$ is irreducible, then

$$Ric\,|_{\mathfrak{m}_j\times\mathfrak{m}_j} = -\frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{l},j}\right) Kill\,|_{\mathfrak{m}_j\times\mathfrak{m}_j},$$

where $c_{\mathfrak{l},j}$ is the eigenvalue of $C_{\mathfrak{l}}$ on $\mathfrak{m}_j$.

Proof: Let $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_b$. If $\mathfrak{m}_1, \ldots, \mathfrak{m}_m$ pairwise commute, then, by Lemma 1.5, we have $Q_{XY}^{\mathfrak{m}_j \mathfrak{m}_k} = 0$, for every $j, k \neq a, b$. In particular, all these bilinear maps vanish if $a \neq b$. Hence, if $a \neq b$, $Ric(X, Y) = -\frac{1}{2}Kill(X, Y) = 0$. Therefore, the Ricci curvature is well determined by elements of the form $Ric(X, X)$, with $X \in \mathfrak{m}_a$, and

$$Ric(X, X) = \frac{1}{4}Q_{XX}^{\mathfrak{m}_a \mathfrak{m}_a} - \frac{1}{2}Kill(X, X).$$

Since $Q_{XX}^{\mathfrak{m}_a \mathfrak{m}_a} = Q_{XX}^{\mathfrak{m}\mathfrak{m}}$, the rest of the proof is similar to the proof of Corollary 1.2.
$\square$

1.2 Homogeneous Riemannian Fibrations

1.2.1 Notation and Hypothesis

In this Section we obtain the Ricci curvature of an invariant metric with totally geodesic fibers on the total space of a homogeneous fibration. We start by settling once and for all the notation used throughout.

Let G be a compact connected semisimple Lie group and $L \subsetneq K \subsetneq G$ connected closed non-trivial subgroups of G. We denote $M = G/L$, $N = G/K$ and $F = K/L$. We consider the natural fibration

$$\begin{array}{rccc} \pi: & M & \to & N \\ & aL & \mapsto & aK \end{array} \tag{1.30}$$

with fiber F and structural group K. We denote by $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{l}$ the Lie algebras of G, K and L, respectively. By $Kill$ we denote the Killing form of G and we set $B = -Kill$. Also, we denote the Killing forms of K and L by $Kill_{\mathfrak{k}}$ and $Kill_{\mathfrak{l}}$, respectively. As G is compact and semisimple, the Killing form of G is negative definite and thus B is positive definite. We consider an orthogonal decomposition of $\mathfrak{g}$ with respect to B given by

$$\mathfrak{g} = \mathfrak{l} \oplus \underbrace{\mathfrak{p} \oplus \mathfrak{n}}_{\mathfrak{m}}, \tag{1.31}$$

where $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$. Clearly, $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$ and $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$ are reductive decompositions for M, N and F, respectively. Hence, we have the following inclusions

$$[\mathfrak{k}, \mathfrak{n}] \subset \mathfrak{n},\ [\mathfrak{l}, \mathfrak{n}] \subset \mathfrak{n},\ [\mathfrak{l}, \mathfrak{p}] \subset \mathfrak{p} \text{ and } [\mathfrak{p}, \mathfrak{n}] \subset \mathfrak{n}. \tag{1.32}$$

An $Ad\,K$-invariant scalar product on $\mathfrak{n}$ induces a G-invariant Riemannian metric g_N on N and an $Ad\,L$-invariant scalar product on $\mathfrak{p}$ induces a G-invariant Riemannian metric g_F on F. The orthogonal direct sum of these scalar products on $\mathfrak{m}$ defines a G-invariant Riemannian metric g_M on M which projects onto a G-invariant metric on N. Moreover, if $\mathfrak{p}$ and $\mathfrak{n}$ do not contain any equivalent $Ad\,L$-submodules, then any G-invariant metric which projects onto a G-invariant metric on N is constructed in this fashion. We recall the following result due to L.Bérard-Bergery ([9], [10] 9 §H):

Theorem 1.3. *The natural projection $\pi : M \to N$ is a Riemannian submersion from (M, g_M) to (N, g_N) with totally geodesic fibers.*

Throughout this thesis we shall refer to such a metric g_M as an adapted metric:

Definition 1.4. *An **adapted** metric on M is a G-invariant Riemannian metric g_M such that the natural projection $\pi : M \to N$ is a Riemannian submersion and consequently the fibers are totally geodesic submanifolds. The fibration $M \to N$ equipped with an adapted metric g is then called a **Riemannian fibration**.*

An adapted metric on M shall be denoted by g_M and, as already introduced above, g_F and g_N shall denote the projection of g_M onto the base space N and g_F its restriction to the fiber F.

We consider a decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_s$ into irreducible $Ad\, L$-submodules pairwise orthogonal with respect to B. Also let $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_n$ be an orthogonal decomposition into irreducible $Ad\, K$-submodules. Throughout we assume the following hypothesis:

(i) $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are pairwise inequivalent irreducible $Ad\, L$-submodules;
(ii) $\mathfrak{n}_1, \ldots, \mathfrak{n}_n$ are pairwise inequivalent irreducible $Ad\, K$-submodules;
(iii) $\mathfrak{p}$ and $\mathfrak{n}$ do not contain equivalent $Ad\, L$-submodules.

We shall refer to this hypothesis by saying that M has **simple spectrum**. Under this hypothesis, according to Schur's Lemma, any $Ad\, L$-invariant scalar product on $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{n}$ which restricts to an $Ad\, K$-invariant scalar product on $\mathfrak{n}$ is of the form

$$<,>= (\oplus_{a=1}^{s} \lambda_a B \mid_{\mathfrak{p}_a \times \mathfrak{p}_a}) \oplus (\oplus_{k=1}^{n} \mu_k B \mid_{\mathfrak{n}_k \times \mathfrak{n}_k}) \,, \tag{1.33}$$

for some $\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_n > 0$. Since an adapted metric g_M on M projects onto a G-invariant Riemannian metric on N, g_M is necessarily induced by a scalar product of the form (1.33). To denote that g_M is induced by a scalar product as in (1.33) we shall write

$$g_M = g_M(\lambda_1, \ldots, \lambda_s;\, \mu_1, \ldots, \mu_n). \tag{1.34}$$

Similarly, we write

$$g_F = g_F(\lambda_1, \ldots, \lambda_s) \text{ and } g_N = g_N(\, \mu_1, \ldots, \mu_n). \tag{1.35}$$

By Ric we mean the Ricci curvature of g_M and by Ric^F and Ric^N the Ricci curvature of g_N and g_F, respectively.

In the following Sections we compute the Ricci curvature for g_M and find some necessary conditions so that g_M is an Einstein metric. We recall that in Theorem 1.2 we have shown that the Ricci curvature of any metric on M can be described using the bilinear maps Q_{XY} defined in 1.2.

1.2.2 The Ricci Curvature in the Direction of the Fiber

In this Section we obtain the Ricci curvature of the adapted metric

$$g_M = g_M(\lambda_1, \ldots, \lambda_s; \, \mu_1, \ldots, \mu_n)$$

in the vertical direction $\mathfrak{p}$. We recall that $\mathfrak{p}$ decomposes into the direct sum of the pairwise inequivalent irreducible $Ad\,L$-submodules $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ and, as explained above, g_M is induced by the scalar product (1.33)

$$(\oplus_{a=1}^{s} \lambda_a B \mid_{\mathfrak{p}_a \times \mathfrak{p}_a}) \oplus (\oplus_{k=1}^{n} \mu_k B \mid_{\mathfrak{n}_k \times \mathfrak{n}_k}),$$

while g_F is the restriction of g_M to the fiber, i.e.,

$$g_F = g_F(\lambda_1, \ldots, \lambda_s).$$

Lemma 1.6. *Let $X \in \mathfrak{p}$ and $Y \in \mathfrak{m}$.*
(i) $Q_{XY}^{\mathfrak{n}_j \mathfrak{p}_a} = Q_{XY}^{\mathfrak{p}_a \mathfrak{n}_j} = 0$, $a = 1, \ldots, s$, $j = 1, \ldots, n$;
(ii) $Q_{XY}^{\mathfrak{n}_i \mathfrak{n}_j} = 0$, if $i \neq j$, $i, j = 1, \ldots, n$;
(iii) $Q_{XY}^{\mathfrak{n}_j \mathfrak{n}_j} = Kill(C_{\mathfrak{n}_j} X, Y)$, $j = 1, \ldots, n$.

Proof: Since $ad_X \mathfrak{p} \subset \mathfrak{k} \perp \mathfrak{n}$ we have $Q_{XY}^{\mathfrak{n}_j \mathfrak{p}_a} = 0$, from Lemma 1.5. From Lemma 1.4 (ii), $Q_{XY}^{\mathfrak{p}_a \mathfrak{n}_j} = Q_{XY}^{\mathfrak{n}_j \mathfrak{p}_a} = 0$.
As $ad_X \mathfrak{n}_j \subset \mathfrak{n}_j$,we have $Q_{XY}^{\mathfrak{n}_j \mathfrak{n}_j} = Kill(C_{\mathfrak{n}_j} X, Y)$. Moreover, since $\mathfrak{n}_j \perp \mathfrak{n}_i$, for every $i \neq j$, we also conclude that $Q_{XY}^{\mathfrak{n}_i \mathfrak{n}_j} = 0$, if $i \neq j$.
□

Lemma 1.7. *The Ricci curvature of $g_F = g_F(\lambda_1, \ldots, \lambda_s)$ is of the form $Ric^F = \oplus_{a=1}^{s} q_a B \mid_{\mathfrak{p}_a \times \mathfrak{p}_a}$, with*

$$q_a = \frac{1}{2} \sum_{b,c=1}^{s} \left(\frac{\lambda_a^2}{2\lambda_c \lambda_b} - \frac{\lambda_c}{\lambda_b} \right) q_a^{cb} + \frac{\gamma_a}{2}$$

where the constants q_a^{cb} and γ_a are such that

$$Kill_{\mathfrak{k}} \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a Kill \mid_{\mathfrak{p}_a \times \mathfrak{p}_a}$$

and

$$Q^{\mathfrak{p}_b \mathfrak{p}_c} \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} = q_a^{cb} Kill \mid_{\mathfrak{p}_a \times \mathfrak{p}_a}.$$

In particular, $Ric^F(\mathfrak{p}_a, \mathfrak{p}_b) = 0$, if $a \neq b$.

Proof: Since $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are pairwise inequivalent irreducible $Ad\,L$-submodules and Ric^F is an $Ad\,L$-invariant symmetric bilinear form, we may write $Ric^F =$

$\oplus_{a=1}^{s} q_a B \mid_{\mathfrak{p}_a \times \mathfrak{p}_a}$, for some constants $q_1, \ldots, q_s$. In particular, we have $Ric^F(\mathfrak{p}_a, \mathfrak{p}_b) = 0$, for every $a, b = 1, \ldots, s$ such that $a \neq b$. By Theorem 1.2, the Ricci curvature of g_F is

$$Ric^F(X,X) = \frac{1}{2} \sum_{b,c=1}^{s} \left(\frac{\lambda_b}{\lambda_c} - \frac{\lambda_a^2}{2\lambda_c \lambda_b} \right) Q_{XX}^{\mathfrak{p}_c \mathfrak{p}_b} - \frac{1}{2} Kill_{\mathfrak{k}}(X,X).$$

By Lemma 1.4 the maps $Q^{\mathfrak{p}_c \mathfrak{p}_b}$ are $Ad\,L$-invariant symmetric bilinear maps. Since $\mathfrak{p}_a$ is $Ad\,L$-irreducible, there are constants q_a^{cb} such that

$$Q^{\mathfrak{p}_c \mathfrak{p}_b} \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} = q_a^{cb} Kill \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} .$$

Similarly, the Killing form of $\mathfrak{k}$, $Kill_{\mathfrak{k}}$, is an $Ad\,L$-invariant symmetric bilinear form on $\mathfrak{p}_a$. So, by irreducibility of $\mathfrak{p}_a$, there is a constant γ_a such that

$$Kill(C_{\mathfrak{k}} \cdot, \cdot) \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a Kill \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} .$$

By the expression above for Ric^F, we must have

$$q_a = \frac{1}{2} \sum_{b,c=1}^{s} \left(\frac{\lambda_a^2}{2\lambda_c \lambda_b} - \frac{\lambda_c}{\lambda_b} \right) q_a^{cb} + \frac{\gamma_a}{2}$$

and the result follows from this.

□

Proposition 1.1. *Let $g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)$ be an adapted metric on M.*

For every $a, b = 1, \ldots, s$ such that $a \neq b$, $Ric(\mathfrak{p}_a, \mathfrak{p}_b) = 0$. For every $X \in \mathfrak{p}_a$, $a = 1, \ldots, s$,

$$Ric(X,X) = \left(q_a + \frac{\lambda_a^2}{4} \sum_{j=1}^{n} \frac{c_{\mathfrak{n}_j,a}}{\mu_j^2} \right) B(X,X),$$

where, for $j = 1, \ldots, n$, the constants $c_{\mathfrak{n}_j,a}$ are such that

$$Kill(C_{\mathfrak{n}_j} \cdot, \cdot) \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} = c_{\mathfrak{n}_j,a} Kill \mid_{\mathfrak{p}_a \times \mathfrak{p}_a}$$

and $C_{\mathfrak{n}_j}$ is the Casimir operator of $\mathfrak{n}_j$ with respect to $Kill$. The constant q_a is such that

$$q_a = \frac{1}{2} \sum_{b,c=1}^{s} \left(\frac{\lambda_a^2}{2\lambda_c \lambda_b} - \frac{\lambda_c}{\lambda_b} \right) q_a^{cb} + \frac{\gamma_a}{2},$$

with q_a^{cb} and γ_a defined by

$$Kill_{\mathfrak{k}}\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}=\gamma_a Kill\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}$$

$$Q^{\mathfrak{p}_b\mathfrak{p}_c}\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}=q_a^{cb}Kill\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}.$$

Proof: Since $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ are pairwise inequivalent irreducible $Ad\,L$-submodules and $Ric\mid_{\mathfrak{p}\times\mathfrak{p}}$ is an $Ad\,L$-invariant symmetric bilinear form, we have that $Ric\mid_{\mathfrak{p}\times\mathfrak{p}}$ is diagonal, i.e.,

$$Ric\mid_{\mathfrak{p}\times\mathfrak{p}}=a_1B\mid_{\mathfrak{p}_1\times\mathfrak{p}_1}\oplus\ldots\oplus a_sB\mid_{\mathfrak{p}_s\times\mathfrak{p}_s},$$

for some constants $a_1,\dots,a_s$. In particular, we have $Ric(\mathfrak{p}_a,\mathfrak{p}_b)=0$, for every $a,b=1,\dots,s$ such that $a\neq b$. Hence, $Ric\mid_{\mathfrak{p}\times\mathfrak{p}}$ is determined by elements $Ric(X,X)$ with $X\in\mathfrak{p}_a$, $a=1,\dots,s$.

By Lemma 1.6 we obtain that only $Q^{\mathfrak{n}_j\mathfrak{n}_j}_{XX}=Kill(C_{\mathfrak{n}_j}X,X)$ and $Q^{\mathfrak{p}_b\mathfrak{p}_c}_{XX}$ are non-zero. Therefore, by Theorem 1.2 we obtain that

$$Ric(X,X)=$$

$$\frac{1}{2}\sum_{j,k=1}^{s}\left(\frac{\lambda_k}{\lambda_j}-\frac{\lambda_a^2}{2\lambda_j\lambda_k}\right)Q^{\mathfrak{p}_j\mathfrak{p}_k}_{XX}+\frac{1}{2}\sum_{j=1}^{n}\left(1-\frac{\lambda_a^2}{2\mu_j^2}\right)Q^{\mathfrak{n}_j\mathfrak{n}_j}_{XX}-\frac{1}{2}Kill(X,X).$$

We have

$$\begin{aligned}\textstyle\sum_{j=1}^m Q^{\mathfrak{n}_j\mathfrak{n}_j}_{XX} &= \textstyle\sum_{j=1}^m Kill(C_{\mathfrak{n}_j}X,Y)\\ &= Kill(C_{\mathfrak{n}}X,X)\\ &= Kill(X,X)-Kill(C_{\mathfrak{k}}X,X)\\ &= Kill(X,X)-Kill_{\mathfrak{k}}(X,X).\end{aligned}$$

Hence we can rewrite $Ric(X,X)$ as follows:

$$\underbrace{\frac{1}{2}\sum_{a,b=1}^{s}\left(\frac{\lambda_b}{\lambda_c}-\frac{\lambda_a^2}{2\lambda_c\lambda_b}\right)Q^{\mathfrak{p}_c\mathfrak{p}_b}_{XX}-\frac{1}{2}Kill_{\mathfrak{k}}(X,X)}_{(1)}-\frac{1}{2}\sum_{j=1}^{n}\frac{\lambda_a^2}{2\mu_j^2}Kill(C_{\mathfrak{n}_j}X,X).$$

As we saw in the proof of Lemma 1.7, the summand (1) is just $Ric^F(X,X)=q_aB(X,X)$.

Furthermore, since $Kill(C_{\mathfrak{n}_j}\cdot,\cdot)=Q^{\mathfrak{n}_j\mathfrak{n}_j}$ are $Ad\,L$-invariant symmetric bilinear maps (Lemma 1.4) and $\mathfrak{p}_a$ is $Ad\,L$-irreducible, there are constants $c_{\mathfrak{n}_j,a}$ such that

$$Kill(C_{\mathfrak{n}_j}\cdot,\cdot)\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}=c_{\mathfrak{n}_j,a}Kill\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}.$$

Therefore,

$$Ric(X,X) = \left(q_a + \frac{1}{4} \sum_{j=1}^{n} \frac{\lambda_a^2}{\mu_j^2} c_{\mathfrak{n}_j,a} \right) B(X,X).$$

□

1.2.3 The Ricci Curvature in the Horizontal Direction

We obtain the Ricci curvature of an adapted metric $g_M = g_M(\lambda_1, \ldots, \lambda_s;\, \mu_1, \ldots, \mu_n)$ in the horizontal direction $\mathfrak{n}$. We recall that $\mathfrak{n}$ decomposes into the direct sum of the pairwise inequivalent irreducible $Ad\, K$-submodules $\mathfrak{n}_1, \ldots, \mathfrak{n}_n$ and, as explained above, g_M is induced by the scalar product (1.33)

$$(\oplus_{a=1}^{s} \lambda_a B \mid_{\mathfrak{p}_a \times \mathfrak{p}_a}) \oplus (\oplus_{k=1}^{n} \mu_k B \mid_{\mathfrak{n}_k \times \mathfrak{n}_k})$$

and g_N is the projection of g_M onto the base space, i.e.,

$$g_N = g_N(\mu_1, \ldots, \mu_n).$$

Lemma 1.8. *Let $X \in \mathfrak{n}_k$ and $Y \in \mathfrak{m}$.*
(i) $Q_{XY}^{\mathfrak{n}_j \mathfrak{p}_a} = Q_{XY}^{\mathfrak{p}_a \mathfrak{n}_j} = 0$, for $j \neq k$;
(ii) $Q_{XY}^{\mathfrak{p}_a \mathfrak{n}_k} = Q_{XY}^{\mathfrak{n}_k \mathfrak{p}_a} = Kill(C_{\mathfrak{p}_a} X, Y)$;
(iii) $Q_{XY}^{\mathfrak{p}_j \mathfrak{p}_a} = 0$, for $i, j = 1, \ldots, s$.

Proof: We have $ad_X \mathfrak{p}_a \subset \mathfrak{n}_k \perp \mathfrak{p}, \mathfrak{n}_j$, for $j \neq k$. Thus, $Q_{XY}^{\mathfrak{n}_j \mathfrak{p}_a} = 0$, for $j \neq k$ and $Q_{XY}^{\mathfrak{p}_j \mathfrak{p}_a} = 0$, for $i, j = 1, \ldots, s$, from Lemma 1.5. Also from $ad_X \mathfrak{p}_a \subset \mathfrak{n}_k$ we deduce that $Q_{XY}^{\mathfrak{p}_a \mathfrak{n}_k} = Kill(C_{\mathfrak{p}_a} X, Y)$. From Lemma 1.4 (ii), we also obtain $Q_{XY}^{\mathfrak{p}_a \mathfrak{n}_j} = Q_{XY}^{\mathfrak{n}_j \mathfrak{p}_a} = 0$, for $j \neq k$ and $Q_{XY}^{\mathfrak{n}_k \mathfrak{p}_a} = Q_{XY}^{\mathfrak{p}_a \mathfrak{n}_k} = Kill(C_{\mathfrak{p}_a} X, Y)$, for $j = k$.
□

Lemma 1.9. *The Ricci curvature of $g_N = g_N(\mu_1, \ldots, \mu_n)$ is of the form $Ric^N = \oplus_{k=1}^{n} r_k B \mid_{\mathfrak{n}_k \times \mathfrak{n}_k}$, where*

$$r_k = \frac{1}{2} \sum_{j,i=1}^{n} \left(\frac{\mu_a^2}{2\mu_i \mu_j} - \frac{\mu_i}{\mu_j} \right) r_k^{ji} + \frac{1}{2}$$

and the constants r_k^{ji} are such that

$$Q^{\mathfrak{n}_j \mathfrak{n}_i} \mid_{\mathfrak{n}_k \times \mathfrak{n}_k} = r_k^{ji} Kill \mid_{\mathfrak{n}_k \times \mathfrak{n}_k} .$$

In particular, $Ric^N(\mathfrak{n}_k, \mathfrak{n}_j) = 0$, if $k \neq j$.

Proof: The metric g_N is induced by an $Ad\,K$-invariant scalar product on $\mathfrak{n}$. Hence, Ric^N is an $Ad\,K$-invariant symmetric bilinear form on $\mathfrak{n}$. Since the subspaces $\mathfrak{n}_j$, $j = 1, \ldots, n$, are irreducible pairwise inequivalent $Ad\,K$-submodules, we may write

$$Ric^N = \oplus_{k=1}^{m} r_k B \mid_{\mathfrak{n}_k \times \mathfrak{n}_k},$$

for some constants $r_1, \ldots, r_n$. It is clear that $Ric^N(\mathfrak{n}_k, \mathfrak{n}_j) = 0$, if $k \neq j$.
It follows from Theorem 1.2 that, for every $X \in \mathfrak{n}_k$,

$$Ric^N(X, X) = \frac{1}{2} \sum_{j,i=1}^{n} \left(\frac{\mu_i}{\mu_j} - \frac{\mu_k^2}{2\mu_i \mu_j} \right) Q_{XX}^{\mathfrak{n}_j \mathfrak{n}_i} - \frac{1}{2} Kill(X, X).$$

Since each subspace $\mathfrak{n}_j$ is $Ad\,K$-invariant, the bilinear maps $Q_{XY}^{\mathfrak{n}_j \mathfrak{n}_i}$ are $Ad\,K$-invariant symmetric bilinear maps (Lemma 1.4). Hence, by irreducibility of the $\mathfrak{n}_k$'s it follows that $Q^{\mathfrak{n}_j \mathfrak{n}_i} \mid_{\mathfrak{n}_k \times \mathfrak{n}_k} = r_k^{ji} Kill \mid_{\mathfrak{n}_k \times \mathfrak{n}_k}$, for some constants r_k^{ji}, $j, i, k = 1, \ldots, n$.

By definition of the r_k's it must be

$$r_k = \frac{1}{2} \sum_{j,i=1}^{n} \left(\frac{\mu_k^2}{2\mu_i \mu_j} - \frac{\mu_i}{\mu_j} \right) r_a^{ji} + \frac{1}{2}.$$

□

Proposition 1.2. *Let $g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)$ be an adapted metric on M. For every $k, j = 1, \ldots, n$, such that $j \neq k$, $Ric(\mathfrak{n}_k, \mathfrak{n}_j) = 0$. For every $X \in \mathfrak{n}_k$,*

$$Ric(X, X) = -\frac{1}{2\mu_k} \sum_{a=1}^{s} \lambda_a B(C_{\mathfrak{p}_a} X, X) + r_k B(X, X),$$

where, for every $a = 1, \ldots, s$, $C_{\mathfrak{p}_a}$ is the Casimir operator of $\mathfrak{p}_a$ with respect to $Kill$,

$$r_k = \frac{1}{2} \sum_{j,i=1}^{n} \left(\frac{\mu_a^2}{2\mu_i \mu_j} - \frac{\mu_i}{\mu_j} \right) r_k^{ji} + \frac{1}{2}$$

and the constants r_k^{ji} are such that

$$Q^{\mathfrak{n}_j \mathfrak{n}_i} \mid_{\mathfrak{n}_k \times \mathfrak{n}_k} = r_k^{ji} Kill \mid_{\mathfrak{n}_k \times \mathfrak{n}_k}.$$

Proof: Let $X \in \mathfrak{n}_k$ and $Y \in \mathfrak{n}_{k'}$. By Lemma 1.8 we have $Q_{XY}^{\mathfrak{p}_a \mathfrak{p}_b} = 0$, for every $a, b = 1, \ldots, s$. Also, $Q_{XY}^{\mathfrak{n}_j \mathfrak{p}_a} = Q_{XY}^{\mathfrak{p}_a \mathfrak{n}_j} = 0$, if $j \neq k, k'$. Therefore, it follows from Theorem 1.2 and Lemma 1.9 that, if $k \neq k'$, then

$$Ric(X,Y) = \frac{1}{2}\sum_{j,i=1}^{n}\left(\frac{\mu_i}{\mu_j} - \frac{\mu_k\mu_{k'}}{2\mu_i\mu_j}\right)Q_{XY}^{\mathfrak{n}_j\mathfrak{n}_i} - \frac{1}{2}Kill(X,Y) = Ric^N(X,Y) = 0.$$

Hence, $Ric\,|_{\mathfrak{n}\times\mathfrak{n}}$ is determined by elements $Ric(X,X)$ with $X \in \mathfrak{n}_k$, $k = 1,\ldots,n$. For $X \in \mathfrak{n}_k$, by Theorem 1.2, we get

$$Ric(X,X) =$$

$$\frac{1}{2}\sum_{k=1}^{s}\left(\frac{\mu_k}{\lambda_a} - \frac{\mu_k^2}{2\mu_k\lambda_a}\right)Q_{XX}^{\mathfrak{p}_a\mathfrak{n}_k} + \frac{1}{2}\sum_{a=1}^{s}\left(\frac{\lambda_a}{\mu_k} - \frac{\mu_k^2}{2\mu_k\lambda_a}\right)Q_{XX}^{\mathfrak{n}_k\mathfrak{p}_a} + Ric^N(X,X).$$

From Lemma 1.8, we know that $Q_{XX}^{\mathfrak{n}_k\mathfrak{p}_a} = Q_{XX}^{\mathfrak{p}_a\mathfrak{n}_k} = Kill(C_{\mathfrak{p}_a}X,X)$. Hence, we simplify the expression above obtaining

$$Ric(X,X) = \frac{1}{2}\sum_{a=1}^{s}\frac{\lambda_a}{\mu_k}Kill(C_{\mathfrak{p}_a}X,X) + Ric^N(X,X).$$

Finally, since $Ric^N(X,X) = r_kB(X,X)$, using the notation of Lemma 1.9, we conclude that

$$Ric(X,X) = -\frac{1}{2}\sum_{a=1}^{s}\frac{\lambda_a}{\mu_k}B(C_{\mathfrak{p}_a}X,X) + r_kB(X,X).$$

□

1.2.4 The Ricci Curvature in the Mixed Direction

In the previous two sections we determined the Ricci curvature of

$$g_M = g_M(\lambda_1,\ldots,\lambda_s;\,\mu_1,\ldots,\mu_n)$$

on the directions of $\mathfrak{p}$ and $\mathfrak{n}$. Here we obtain the Ricci curvature in the direction of $\mathfrak{p}\times\mathfrak{n}$.

Proposition 1.3. *Let $g_M = g_M(\lambda_1,\ldots,\lambda_s;\,\mu_1,\ldots,\mu_n)$ be an adapted metric on M. For every $X \in \mathfrak{p}_a$ and $Y \in \mathfrak{n}_k$,*

$$Ric(X,Y) = \frac{\lambda_a\mu_k}{4}\sum_{j=1}^{n}\frac{B(C_{\mathfrak{n}_j}X,Y)}{\mu_j^2},$$

where, for every $j = 1,\ldots,n$, $C_{\mathfrak{n}_j}$ is the Casimir operator of $\mathfrak{n}_j$ with respect to $Kill$.

Proof: For $X \in \mathfrak{p}$ we know from Lemma 1.6 that $Q_{XY}^{\mathfrak{n}_j\mathfrak{p}_a} = Q_{XY}^{\mathfrak{p}_a\mathfrak{n}_j} = 0$, for every $a = 1, \ldots, s$, $j = 1, \ldots, n$, and $Q_{XY}^{\mathfrak{n}_i\mathfrak{n}_j} = 0$, if $i \neq j$, $i, j = 1, \ldots, n$, whereas $Q_{XY}^{\mathfrak{n}_j\mathfrak{n}_j} = Kill(C_{\mathfrak{n}_j}X, Y)$, $j = 1, \ldots, n$. Moreover, for $Y \in \mathfrak{n}_k$, since $ad_X ad_Y \mathfrak{p} \subset \mathfrak{n}_k \perp \mathfrak{p}$, from Lemma 1.6 we also obtain that $Q_{XY}^{\mathfrak{p}_b\mathfrak{p}_c} = 0$, for every $b, c = 1, \ldots, s$. Therefore, only $Q_{XY}^{\mathfrak{n}_j\mathfrak{n}_j} = Kill(C_{\mathfrak{n}_j}X, Y)$, $j = 1, \ldots, n$, may not be zero. Furthermore, $Kill(X, Y) = 0$. Hence, from Theorem 1.2 we get

$$Ric(X,Y) = \frac{1}{2}\sum_{j=1}^{n}\left(1 - \frac{\lambda_a\mu_k}{\mu_j^2}\right) Q_{XY}^{\mathfrak{n}_j\mathfrak{n}_j} = \frac{1}{2}\sum_{j=1}^{n}\left(1 - \frac{\lambda_a\mu_k}{\mu_j^2}\right) Kill(C_{\mathfrak{n}_j}X, Y).$$

On the other hand,

$$\sum_{j=1}^{n} Kill(C_{\mathfrak{n}_j}X, Y) = Kill(C_{\mathfrak{n}}X, Y) = Kill(X, Y) - Kill(C_{\mathfrak{k}}X, Y) = 0,$$

since $C_{\mathfrak{k}}\mathfrak{p} \subset \mathfrak{k} \perp \mathfrak{n}$. Therefore,

$$Ric(X,Y) = -\frac{\lambda_a\mu_k}{4}\sum_{j=1}^{n}\frac{Kill(C_{\mathfrak{n}_j}X, Y)}{\mu_j^2}.$$

□

1.2.5 Necessary Conditions for the Existence of an Adapted Einstein Metric

From the expressions obtained previously for the Ricci curvature in the horizontal direction and in the direction of $\mathfrak{p} \times \mathfrak{n}$ we obtain two necessary conditions for the existence of an adapted Einstein metric on M. These are restrictions on Casimir operators and shall be extremely useful in the chapters ahead.

Corollary 1.4. *Let $g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)$ be an adapted metric on M. If g_M is Einstein, then the operator $\sum_{a=1}^{s} \lambda_a C_{\mathfrak{p}_a} |_{\mathfrak{n}_k}$ is scalar.*

Proof: Let g_M be an adapted metric as defined in (1.33). If g_M is Einstein with Einstein constant E, then, $Ric\ |_{\mathfrak{n}\times\mathfrak{n}} = E <,>|_{\mathfrak{n}\times\mathfrak{n}}$ and thus $Ric\ |_{\mathfrak{n}\times\mathfrak{n}}$ is $Ad\,K$-invariant. Therefore, by Proposition 1.2, we conclude that $\sum_{a=1}^{s} \lambda_a C_{\mathfrak{p}_a} |_{\mathfrak{n}}$ has to be $Ad\,K$-invariant. Hence, $\sum_{a=1}^{s} \lambda_a C_{\mathfrak{p}_a} |_{\mathfrak{n}_k}$ is scalar, by irreducibility of $\mathfrak{n}_k$.
□

Corollary 1.5. *Let $g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)$ be an adapted metric on M. The orthogonality condition $Ric(\mathfrak{p}, \mathfrak{n}) = 0$ holds if and only if*

$$\sum_{j=1}^{n} \frac{1}{\mu_j^2} C_{\mathfrak{n}_j}(\mathfrak{p}) \subset \mathfrak{k}. \tag{1.36}$$

Moreover, if g_M is Einstein, then (1.36) holds.

Proof: Let g_M be an adapted metric of the form $g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)$. From Proposition 1.3, we obtain that $Ric(\mathfrak{p}, \mathfrak{n}) = 0$ if and only if, for every $X \in \mathfrak{p}_a$ and $Y \in \mathfrak{n}_b$,

$$Kill\left(\sum_{j=1}^{n} \frac{C_{\mathfrak{n}_j}}{\mu_j^2} X, Y\right) = 0.$$

This holds if only if $\sum_{j=1}^{n} \frac{C_{\mathfrak{n}_j}}{\mu_j^2} X \subset \mathfrak{k}$, for every $X \in \mathfrak{p}$.
If g_M is Einstein with Einstein constant E, then $Ric(\mathfrak{p}, \mathfrak{n}) = E < \mathfrak{p}, \mathfrak{n} >= 0$. This shows the last assertion of the Corollary.

□

The two previous Corollaries may be restated in the following way, which emphasizes the fact that the two necessary conditions obtained for existence of an Einstein adapted metric are just algebraic conditions on the Casimir operators of the submodules $\mathfrak{p}_a$ and $\mathfrak{n}_k$.

Corollary 1.6. *If there exists on M an Einstein adapted metric, then there are positive constants $\lambda_1, \ldots, \lambda_s$ such that the operator $\sum_{a=1}^{s} \lambda_a C_{\mathfrak{p}_a} |_{\mathfrak{n}_k}$ is scalar. Furthermore, if g_N is not the standard metric, then there are positive constants $\nu_1, \ldots, \nu_n$, not all equal, such that*

$$\sum_{j=1}^{n} \nu_j C_{\mathfrak{n}_j}(\mathfrak{p}) \subset \mathfrak{k}.$$

Proof: The assertions follow from Corollaries 1.4 and 1.5. In 1.5 we set $\nu_k = 1/\mu_k^2$. Hence, $\nu_1 = \ldots = \nu_n$ occurs when g_N is the standard metric. Moreover, if $\nu_1 = \ldots = \nu_n$, the inclusion in 1.5 is equivalent to $C_{\mathfrak{n}}(\mathfrak{p}) \subset \mathfrak{k}$, which always holds since $C_{\mathfrak{n}} = Id - C_{\mathfrak{k}}$ and $C_{\mathfrak{k}}$ maps $\mathfrak{p}$ into $\mathfrak{k}$. So we obtain a condition on the $C_{\mathfrak{n}_j}$'s only when g_N is not standard.

□

CHAPTER 2

As in Chapter 1, we consider a homogeneous fibration $F \to M \to N$, for $M = G/L$, $N = G/K$ and $F = K/L$, where G is a compact connected semisimple Lie group and $L \subsetneq K \subsetneq G$ are connected closed non-trivial subgroups, and an adapted metric g_M on M. We consider some particular cases by imposing restrictions on the metric g_M, which shall lead to simpler expressions of the Ricci curvature and thus allow us to determine further conditions for the existence of an Einstein adapted metric. Unless otherwise stated we follow the notation used in Chapter 1. Thus, as before, $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are the irreducible pairwise inequivalent $Ad\,L$-submodules of $\mathfrak{p}$, the tangent space to the fiber, and $\mathfrak{n}_1, \ldots, \mathfrak{n}_n$ are the irreducible pairwise inequivalent $Ad\,K$-submodules of $\mathfrak{n}$, the tangent space to the base. An adapted metric g_M on M is written as

$$g_M = g_M(\lambda_1, \ldots, \lambda_s;\, \mu_1, \ldots, \mu_n)$$

meaning that g_M is induced by the scalar product

$$(\oplus_{a=1}^{s} \lambda_a B\mid_{\mathfrak{p}_a \times \mathfrak{p}_a}) \oplus (\oplus_{k=1}^{n} \mu_k B\mid_{\mathfrak{n}_k \times \mathfrak{n}_k}),$$

on the tangent space $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{n}$ of M. We assume that M has simple spectrum as in Section 1.2.1.

2.1 Riemannian Fibrations with Normal Fiber

In this section we consider an adapted metric g_M whose restriction to the fiber, g_F, is a multiple of the Killing form of $\mathfrak{g}$. Hence, we have

$$g_M = g_M(\underbrace{\lambda, \ldots, \lambda}_{s};\, \mu_1, \ldots, \mu_n) \tag{2.1}$$

and

$$g_F = g_F(\underbrace{\lambda, \ldots, \lambda}_{s}) \tag{2.2}$$

by setting $\lambda_1 = \ldots = \lambda_s = \lambda$ in (1.33) and (1.34). Clearly, when equipped with g_F, F becomes a normal Riemannian manifold. In particular, if the Killing form of $\mathfrak{k}$ is a multiple of the Killing form of $\mathfrak{g}$, then F is a standard Riemannian manifold. This shall be the case when, for instance, $\mathfrak{p}$ is irreducible, but it will not be the case in general.

Proposition 2.1. *Let g_M be an adapted metric on M of the form*

$$g_M(\underbrace{\lambda,\ldots,\lambda}_{s};\,\mu_1,\ldots,\mu_n).$$

The Ricci curvature of g_M is as follows:

(i) For every $X\in\mathfrak{p}_a$,

$$Ric(X,X)=\left(q_a+\frac{\lambda^2}{4}\sum_{j=1}^{n}\frac{c_{\mathfrak{n}_j,a}}{\mu_j^2}\right)B(X,X),$$

with

$$q_a=\frac{1}{2}\left(c_{\mathfrak{l},a}+\frac{\gamma_a}{2}\right),$$

where $c_{\mathfrak{l},a}$ is the eigenvalue of the Casimir operator of $\mathfrak{l}$ on $\mathfrak{p}_a$, γ_a is defined by

$$Kill_{\mathfrak{k}}\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}=\gamma_a Kill\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}$$

and $c_{\mathfrak{n}_j,a}$ is given by

$$Kill(C_{\mathfrak{n}_j}\cdot,\cdot)\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}=c_{\mathfrak{n}_j,a}Kill\mid_{\mathfrak{p}_a\times\mathfrak{p}_a},$$

where $C_{\mathfrak{n}_j}$ is the Casimir operator of $\mathfrak{n}_j$ with respect to $Kill$.
(ii) For every $X\in\mathfrak{n}_k$,

$$Ric(X,X)=-\frac{\lambda}{2\mu_k}B(C_{\mathfrak{p}}X,X)+r_kB(X,X),$$

where r_k is as defined in Lemma 1.9;
(iii) For every $X\in\mathfrak{p}_a$ and $Y\in\mathfrak{n}_k$,

$$Ric(X,Y)=\frac{\lambda\mu_k}{4}\sum_{j=1}^{n}\frac{B(C_{\mathfrak{n}_j}X,Y)}{\mu_j^2};$$

(iv) $Ric(\mathfrak{p}_a,\mathfrak{p}_b)=0$, for every $a,b=1,\ldots,s$ such that $a\neq b$, and $Ric(\mathfrak{n}_i,\mathfrak{n}_j)=0$, for every $i,j=1,\ldots,n$ such that $i\neq j$.

Proof: (i) By Corollary 1.2, if g_F is a multiple of B, then we obtain that

$$Ric^F(X,X)=-\frac{1}{2}\left(\frac{1}{2}+c'_{\mathfrak{l},a}\right)Kill_{\mathfrak{k}}(X,X),$$

for $X \in \mathfrak{p}_a$, where $c'_{\mathfrak{l},a}$ is the eigenvalue of the Casimir operator of $\mathfrak{l}$ with respect to the Killing form of $\mathfrak{k}$ on $\mathfrak{p}_a$. Clearly,

$$\begin{aligned} c'_{\mathfrak{l},a} Kill_{\mathfrak{k}}(X,X) = & \; Kill_{\mathfrak{k}}(C'_{\mathfrak{l}}X,X) \\ = & \; tr(ad_X^2 \mid_{\mathfrak{p}_a}) \\ = & \; Kill(C_{\mathfrak{l}}X,X) \\ = & \; c_{\mathfrak{l},a} Kill(X,X). \end{aligned}$$

By recalling that
$Kill_{\mathfrak{k}} \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a Kill \mid_{\mathfrak{p}_a \times \mathfrak{p}_a}$, we write $Ric^F = \frac{1}{2}\left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a}\right) B$ and by following the notation in Lemma 1.7, we have

$$q_a = \frac{1}{2}\left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a}\right).$$

The result then follows from this and Proposition 1.1.

(ii) follows directly from Proposition 1.2, by observing that $\sum_{a=1}^{s} C_{\mathfrak{p}_a} = C_{\mathfrak{p}}$.
(iii) The Ricci curvature in the direction $\mathfrak{p} \times \mathfrak{n}$ essentially remains unchanged; the expression given is just that of Proposition 1.3 after replacing $\lambda_1, \ldots, \lambda_s$ by λ.
(iv) These orthogonality conditions are satisfied by any adapted metric on M and were shown to hold in Propositions 1.1 and 1.2.
□

Proposition 2.2. *Let $g_M(\underbrace{\lambda, \ldots, \lambda}_{s}; \mu_1, \ldots, \mu_n)$ be any adapted metric on M and suppose that $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ pairwise commute, i.e., $[\mathfrak{p}_a, \mathfrak{p}_b] = 0$ if $a \neq b$. For every $X \in \mathfrak{p}_a$,*

$$Ric(X,X) = \left(q_a + \frac{\lambda_a^2}{4} \sum_{j=1}^{n} \frac{c_{\mathfrak{n}_j,a}}{\mu_j^2}\right) B(X,X),$$

where all the constants are as in Proposition 2.1.

Proof: This follows from the fact that, in this case, Corollary 1.3 gives also

$$Ric^F(X,X) = -\frac{1}{2}\left(\frac{1}{2} + c'_{\mathfrak{l},a}\right) Kill_{\mathfrak{k}}(X,X),$$

as in the previous proof. Hence, the expression for $Ric(X,X)$ is exactly the same obtained above for (i) in Proposition 2.1.
□

Corollary 2.1. *If there exists on M an Einstein adapted metric of the form $g_M(\underbrace{\lambda, \ldots, \lambda}_{s}; \mu_1, \ldots, \mu_n)$, then $C_{\mathfrak{p}}$ and $C_{\mathfrak{l}}$ are scalar on $\mathfrak{n}_k$, $k = 1, \ldots, n$.*

Proof: Since $\sum_{a=1}^{s} C_{\mathfrak{p}_a} = C_{\mathfrak{p}}$, the necessary condition for g_M to be Einstein given in Corollary 1.4, translates into the condition that $C_{\mathfrak{p}}$ is scalar on $\mathfrak{n}_j$, if $\lambda_1 = \ldots = \lambda_s = \lambda$. We have that $C_{\mathfrak{k}} = C_{\mathfrak{p}} + C_{\mathfrak{l}}$ is scalar on $\mathfrak{n}_j$, since $\mathfrak{n}_j$ is irreducible as a K-module. Then $C_{\mathfrak{p}}$ is scalar on $\mathfrak{n}_j$ if and only if $C_{\mathfrak{l}}$ is.
□

2.2 Riemannian Fibrations with Standard Base

In this section we consider an adapted metric g_M whose projection onto the base space, g_N, is a multiple of the Killing form of $\mathfrak{g}$. Hence, we have

$$g_M = g_M(\lambda_1, \ldots, \lambda_s; \underbrace{\mu, \ldots, \mu}_{n}) \tag{2.3}$$

and

$$g_N = g_N(\underbrace{\mu, \ldots, \mu}_{n}), \tag{2.4}$$

by setting $\mu_1 = \ldots = \mu_n = \mu$ in (1.33) and (1.34). In this particular case, when equipped with g_N, N is a standard Riemannian manifold.

Proposition 2.3. *Let g_M be an adapted metric on M of the form*

$$g_M(\lambda_1, \ldots, \lambda_s; \underbrace{\mu, \ldots, \mu}_{n}).$$

The Ricci curvature of g_M is as follows:
(i) For every $X \in \mathfrak{p}_a$,

$$Ric(X, X) = \left(q_a + \frac{\lambda_a^2}{4\mu^2}(1 - \gamma_a)\right) B(X, X),$$

where q_a and γ_a are as defined in Lemma 1.7, i.e., they are defined by the identities

$$Kill_{\mathfrak{k}} \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a Kill \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} \text{ and } Ric^F \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} = q_a B \mid_{\mathfrak{p}_a \times \mathfrak{p}_a};$$

(ii) For every $X \in \mathfrak{n}_k$,

$$Ric(X, X) = -\frac{1}{2} \sum_{a=1}^{s} \frac{\lambda_a}{\mu} B(C_{\mathfrak{p}_a} X, X) + r_k B(X, X),$$

with

$$r_k = \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{k},k}\right),$$

where $c_{\mathfrak{k},k}$ is the eigenvalue of the Casimir operator $C_{\mathfrak{k}}$ on $\mathfrak{n}_k$;
(iii) $Ric(\mathfrak{p}, \mathfrak{n}) = 0$;
(iv) $Ric(\mathfrak{p}_a, \mathfrak{p}_b) = 0$, for every $a, b = 1, \ldots, s$ such that $a \neq b$, and $Ric(\mathfrak{n}_i, \mathfrak{n}_j) = 0$, for every $i, j = 1, \ldots, n$ such that $i \neq j$.

Proof: (i) From the fact that $\gamma_a + \sum_{j=1}^{n} c_{\mathfrak{n}_j,a} = 1$, we obtain

$$\sum_{j=1}^{n} \frac{\lambda_a^2}{\mu^2} c_{\mathfrak{n}_j,a} = \frac{\lambda_a^2}{\mu^2}(1-\gamma_a).$$

The required expression follows immediately from Proposition 1.1.

(ii) From Corollary 1.2 we obtain that $r_k = \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{k},k}\right)$, where r_k is as defined in Lemma 1.9. The expression then follows from Proposition 1.2.

(iii) By using the fact that $C_{\mathfrak{n}} = \sum_{j=1}^{n} C_{\mathfrak{n}_j}$, from Proposition 1.3 it follows that

$$Ric(X,Y) = \frac{\lambda_a}{4\mu} B(C_{\mathfrak{n}}X, Y),$$

for every $X \in \mathfrak{p}_a$ and $Y \in \mathfrak{n}_k$. Moreover, since $C_{\mathfrak{n}} = C_{\mathfrak{g}} - C_{\mathfrak{k}} = Id - C_{\mathfrak{k}}$ and $C_{\mathfrak{k}}(\mathfrak{p}) \subset \mathfrak{k}$, we have that $C_{\mathfrak{n}}(X) \in \mathfrak{k}$ is orthogonal to $Y \in \mathfrak{n}$ with respect to B. Hence, $Ric(X,Y) = 0$.

(iv) these orthogonality conditions are simply those in Propositions 1.1 and 1.2.
□

Corollary 2.2. *Let g_M be any adapted metric on M and suppose that $\mathfrak{n}_1, \ldots, \mathfrak{n}_n$ pairwise commute, i.e., $[\mathfrak{n}_j, \mathfrak{n}_k] = 0$, for $k \neq j$. Then, for every $X \in \mathfrak{n}_k$,*

$$Ric(X,X) = -\frac{1}{2}\sum_{a=1}^{s} \frac{\lambda_a}{\mu_k} B(C_{\mathfrak{p}_a}X, X) + r_k B(X,X),$$

where $r_k = \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{k},k}\right)$ and $c_{\mathfrak{k},k}$ is the eigenvalue of the Casimir operator $C_{\mathfrak{k}}$ on $\mathfrak{n}_k$.

Proof: The proof is immediate by using Corollary 1.3 and Proposition 1.2.
□

2.3 Binormal Riemannian Fibrations

A G-invariant metric g_M on M of the form

$$g_M = (\underbrace{\lambda, \ldots, \lambda}_{s}; \underbrace{\mu, \ldots, \mu}_{n}) \tag{2.5}$$

is called **binormal**. That is, a binormal metric is induced by the scalar product

$$\lambda B\mid_{\mathfrak{p}\times\mathfrak{p}} \oplus \mu B\mid_{\mathfrak{n}\times\mathfrak{n}}$$

on $\mathfrak{m}$. The fibration $F \to M \to N$ is then called a **binormal Riemannian fibration**. Clearly, a binormal metric projects onto an invariant metric on the base space N and thus it is an adapted metric. For a binormal metric g_M, both g_N and g_F are multiples of the Killing form of $\mathfrak{g}$

$$g_F = (\underbrace{\lambda, \ldots, \lambda}_{s}) \text{ and } g_N = (\underbrace{\mu, \ldots, \mu}_{n}) \tag{2.6}$$

and thus F is a normal Riemannian manifold, which is standard if $Kill_{\mathfrak{k}}$ is a multiple of $Kill$, and N is a standard Riemannian manifold.

In this Section we obtain the Ricci curvature of a binormal metric g_M on M and conditions for such a metric to be Einstein. As we shall see, the conditions for the existence of an Einstein binormal metric translate in very simple conditions on the Casimir operators of $\mathfrak{k}$, $\mathfrak{l}$ and $\mathfrak{p}_a$, $a = 1, \ldots, s$. The results that we found in Sections 2.1 and 2.2 yield the following description of the Ricci curvature:

Corollary 2.3. *Let $g_M = g_M(\underbrace{\lambda, \ldots, \lambda}_{s}; \underbrace{\mu, \ldots, \mu}_{n})$ be a binormal metric on M.*

(i) For every $X \in \mathfrak{p}_a$,

$$Ric(X, X) = \left(q_a + \frac{\lambda^2}{4\mu^2}(1 - \gamma_a) \right) B(X, X),$$

where $q_a = \frac{1}{2}\left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a}\right)$, $c_{\mathfrak{l},a}$ is the eigenvalue of $C_{\mathfrak{l}}$ on $\mathfrak{p}_a$ and γ_a is determined by

$$Kill_{\mathfrak{k}} \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a Kill \mid_{\mathfrak{p}_a \times \mathfrak{p}_a};$$

(ii) For every $Y \in \mathfrak{n}_j$,

$$Ric(Y, Y) = -\frac{\lambda}{2\mu} B(C_{\mathfrak{p}} Y, Y) + r_j B(Y, Y),$$

where $r_j = \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{k},j}\right)$ and $c_{\mathfrak{k},j}$ is the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{n}_j$;
(iii) Moreover, $Ric(\mathfrak{p}, \mathfrak{n}) = 0$;
(iv) $Ric(\mathfrak{p}_a, \mathfrak{p}_b) = 0$, for every $a, b = 1, \ldots, s$ such that $a \neq b$, and $Ric(\mathfrak{n}_i, \mathfrak{n}_j) = 0$, for every $i, j = 1, \ldots, n$ such that $i \neq j$.

Proof: For a binormal metric on M, g_F and g_N are multiples of $Kill$, so we use Propositions 2.1 and 2.3.
□

Definition 2.1. *For $i, j = 1, \ldots, n$ and $a, b = 1, \ldots, s$, we set*
(i) $\delta^{\mathfrak{k}}_{ij} = c_{\mathfrak{k},i} - c_{\mathfrak{k},j}$ and $\delta^{\mathfrak{l}}_{ij} = c_{\mathfrak{l},i} - c_{\mathfrak{l},j}$;
(ii) $\delta^{\mathfrak{k}}_{ab} = \gamma_a - \gamma_b$ and $\delta^{\mathfrak{l}}_{ab} = c_{\mathfrak{l},a} - c_{\mathfrak{l},b}$.

Theorem 2.1. *(i) If $C_{\mathfrak{p}}$ is not scalar on each $\mathfrak{n}_j$, then there are no binormal Einstein metrics on M;*
(ii) Suppose that $C_{\mathfrak{p}}$ is scalar on each $\mathfrak{n}_j$ and write $C_{\mathfrak{p}} |_{\mathfrak{n}_j}= b^j Id_{\mathfrak{n}_j}$, for $j = 1, \ldots, n$. Then there is a one-to-one correspondence, up to homothety, between binormal Einstein metrics on M and positive solutions of the following set of equations on the unknown X:

$$\delta^{\mathfrak{k}}_{ij}(1 - X) = \delta^{\mathfrak{l}}_{ij}, \text{ if } n > 1, \tag{2.7}$$

$$(2\delta^{\mathfrak{l}}_{ab} + \delta^{\mathfrak{k}}_{ab})X^2 = \delta^{\mathfrak{k}}_{ab}, \text{ if } s > 1, \tag{2.8}$$

$$(\gamma_a + 2c_{\mathfrak{l},a})\, X^2 - (1 + 2c_{\mathfrak{k},j})\, X + (1 - \gamma_a + 2b^j) = 0. \tag{2.9}$$

for every $a, b = 1, \ldots, s$ and $i, j = 1, \ldots, n$, where $c_{\mathfrak{l},a}$ is the eigenvalue of $C_{\mathfrak{l}}$ on $\mathfrak{p}_a$, γ_a is determined by

$$Kill_{\mathfrak{k}} |_{\mathfrak{p}_a\times\mathfrak{p}_a}= \gamma_a Kill |_{\mathfrak{p}_a\times\mathfrak{p}_a},$$

$c_{\mathfrak{k},j}$ is the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{n}_j$ and the δ's are as in Definition 2.1. If such a positive solution X exists, then binormal Einstein metrics are, up to homothety, given by

$$<,>= B |_{\mathfrak{p}\times\mathfrak{p}} \oplus XB |_{\mathfrak{n}\times\mathfrak{n}} .$$

Proof: Let $g_M(\underbrace{\lambda, \ldots, \lambda}_{s}; \underbrace{\mu, \ldots, \mu}_{n})$ be a binormal metric on M and

$$X = \frac{\mu}{\lambda}.$$

By Lemma 2.1, we have that, if g_M is Einstein, then $C_{\mathfrak{p}}$ and $C_{\mathfrak{l}}$ are scalar on $\mathfrak{n}_j$, for every $j = 1, \ldots, n$. Say

$$C_{\mathfrak{p}} |_{\mathfrak{n}_j}= b^j Id \text{ and } C_{\mathfrak{l}} |_{\mathfrak{n}_j}= c_{\mathfrak{l},j} Id.$$

Suppose that g is Einstein with constant E. From Corollary 2.3, we obtain the Einstein equations

$$-\frac{\lambda}{2\mu} b^j + r_j = \mu E,\ j = 1, \ldots, n \tag{2.10}$$

$$\frac{1}{2}\left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a} + \frac{\lambda^2}{2\mu^2}(1 - \gamma_a)\right) = \lambda E,\ a = 1, \ldots, s. \tag{2.11}$$

If $n > 1$, from Equation (2.10) we obtain the following:

$$\frac{\lambda}{2\mu}(b^i - b^j) = r_i - r_j,\ i, j = 1, \ldots, n. \tag{2.12}$$

By using Lemma 1.9 we have

$$r_i - r_j = \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{k},i}\right) - \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{k},j}\right) = \frac{1}{2}(c_{\mathfrak{k},i} - c_{\mathfrak{k},j}),$$

whereas

$$b^i - b^j = (c_{\mathfrak{k},i} - c_{\mathfrak{k},j}) - (c_{\mathfrak{l},i} - c_{\mathfrak{l},j}).$$

Therefore, Equation (2.12) becomes

$$-\frac{\lambda}{\mu}\underbrace{(c_{\mathfrak{l},i} - c_{\mathfrak{l},j})}_{\delta^{\mathfrak{l}}_{ij}} = \left(1 - \frac{\lambda}{\mu}\right)\underbrace{(c_{\mathfrak{k},i} - c_{\mathfrak{k},j})}_{\delta^{\mathfrak{k}}_{ij}}.$$

By using the variable X, we rewrite the equation above as $-\frac{1}{X}\delta^{\mathfrak{l}}_{ij} = \left(1 - \frac{1}{X}\right)\delta^{\mathfrak{k}}_{ij}$, and this yields $\delta^{\mathfrak{l}}_{ij} = (1 - X)\delta^{\mathfrak{k}}_{ij}$.

Equation (2.11) may be rewritten as

$$\frac{1}{2}\left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a}\right)X + (1 - \gamma_a)\frac{1}{4X} = \mu E. \tag{2.13}$$

Hence, if $s > 1$, for $a, b = 1, \ldots, s$, we get

$$\frac{1}{2}\left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a}\right)X + (1 - \gamma_a)\frac{1}{4X} = \frac{1}{2}\left(\frac{\gamma_b}{2} + c_{\mathfrak{l},b}\right)X + (1 - \gamma_b)\frac{1}{4X},$$

which yields

$$\underbrace{c_{\mathfrak{l},a} - c_{\mathfrak{l},b}}_{\delta^{\mathfrak{l}}_{ab}} = \frac{1}{2}\left(\frac{1}{X^2} - 1\right)\underbrace{(\gamma_a - \gamma_b)}_{\delta^{\mathfrak{k}}_{ab}}.$$

By solving this equation we obtain

$$(2\delta^{\mathfrak{l}}_{ab} + \delta^{\mathfrak{k}}_{ab})X^2 = \delta^{\mathfrak{k}}_{ab}.$$

Finally, by using Equations (2.10) and (2.13) we obtain the equality

$$\frac{1}{2}\left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a}\right)X + (1 - \gamma_a)\frac{1}{4X} = -\frac{b^j}{2X} + \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{k},j}\right),$$

which rearranged gives

$$\left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a}\right)X^2 - \left(\frac{1}{2} + c_{\mathfrak{k},j}\right)X + \frac{1}{2}(1 - \gamma_a + 2b^j) = 0.$$

□

An immediate Corollary is the following:

Corollary 2.4. *Suppose that F and N are isotropy irreducible spaces such that $dim\, F > 1$. There exists on M an Einstein adapted metric if and only if $C_{\mathfrak{p}}$ is scalar on $\mathfrak{n}$ and $\triangle \geq 0$, where*

$$\triangle = (1 + 2c_{\mathfrak{k},\mathfrak{n}})^2 - 4(\gamma + 2c_{\mathfrak{l},\mathfrak{p}})(1 - \gamma + 2b),$$

$C_{\mathfrak{k},\mathfrak{n}}$ is the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{n}$, $C_{\mathfrak{l},\mathfrak{p}}$ is the eigenvalue of $C_{\mathfrak{l}}$ on $\mathfrak{p}$, b is the eigenvalue of $C_{\mathfrak{p}}$ on $\mathfrak{n}$ and γ is such that $Kill_{\mathfrak{k}} \mid_{\mathfrak{p}\times\mathfrak{p}} = \gamma Kill \mid_{\mathfrak{p}\times\mathfrak{p}}$.
If all these conditions are satisfied, then Einstein adapted metrics are, up to homothety, given by

$$g_M = B \mid_{\mathfrak{p}\times\mathfrak{p}} \oplus XB \mid_{\mathfrak{n}\times\mathfrak{n}}, \text{ where } X = \frac{1 + 2c_{\mathfrak{k},\mathfrak{n}} \pm \sqrt{\triangle}}{2(\gamma + 2c_{\mathfrak{l},\mathfrak{p}})}.$$

Proof: Since $\mathfrak{p}$ is an irreducible $Ad\, L$-module and $\mathfrak{n}$ is an irreducible $Ad\, K$-module, then any adapted metric on M is binormal. Hence, we use Theorem 2.1. By the irreducibility of $\mathfrak{p}$ and $\mathfrak{n}$, we have $s = 1$ and $n = 1$ and thus Einstein binormal metrics are given by positive solutions of (2.9), if $C_{\mathfrak{p}}$ is scalar on $\mathfrak{n}$. Hence, from Theorem 2.1 we conclude that there exists on M an Einstein binormal metric if and only if $C_{\mathfrak{p}}$ is scalar on $\mathfrak{n}$ and $\triangle \geq 0$, where

$$\triangle = (1 + 2c_{\mathfrak{k},\mathfrak{n}})^2 - 4(\gamma + 2c_{\mathfrak{l},\mathfrak{p}})(1 - \gamma + 2b).$$

Since F is isotropy irreducible and $dim\, F > 1$, we have $\gamma + 2c_{\mathfrak{l},p} \neq 0$ and the polynomial in (2.9) has exactly degree two. In fact, if $\gamma + 2c_{\mathfrak{l},p} = 0$, then $\gamma = c_{\mathfrak{l},p} = 0$ and thus, in particular, $\mathfrak{p}$ lies in the center of $\mathfrak{k}$. But the hypothesis that $\mathfrak{p}$ is irreducible and abelian implies that $\mathfrak{p}$ is 1-dimensional which contradicts the hypothesis that $dim\, F > 1$. Therefore, $\gamma + 2c_{\mathfrak{l},p} \neq 0$. In this case, the solutions of (2.9) are

$$X = \frac{1 + 2c_{\mathfrak{k},\mathfrak{n}} \pm \sqrt{\triangle}}{2(\gamma + 2c_{\mathfrak{l},\mathfrak{p}})}.$$

□

In the case when F is 1-dimensional, the fibration $M \to N$ is a principal circle bundle, since F is an abelian compact connected 1-dimensional group. We recall that Einstein metrics on principal fiber bundles have been widely studied ([19],[46]) and, in particular, homogeneous Einstein metrics on circle bundles were classified McKenzie Y. Wang and Wolfgang Ziller in [46]. We revisit metrics on circle bundles by stating the following:

Corollary 2.5. *Suppose that N is isotropy irreducible and F is isomorphic to the circle group. There exists on M exactly one G-invariant Einstein metric, up to homothety, given by*

$$g_M = B\mid_{\mathfrak{p}\times\mathfrak{p}} \oplus XB\mid_{\mathfrak{n}\times\mathfrak{n}},\ with\ X = \frac{2+m}{m(1+2c_{\mathfrak{k},\mathfrak{n}})},$$

where $c_{\mathfrak{k},\mathfrak{n}}$ *is the eigenvalue of* $C_{\mathfrak{k}}$ *on* $\mathfrak{n}$ *and* $m = dim\, G/K$.

Proof: The fact that $\mathfrak{p}$ is 1-dimensional implies that $\mathfrak{p}$ lies in the center of $\mathfrak{k}$. Hence, in the notation of Corollary 2.4, $\gamma = c_{\mathfrak{l},\mathfrak{p}} = 0$. On the other hand, if $\mathfrak{n}$ is $Ad\,K$-irreducible then, the semisimple part of K acts transitively on $\mathfrak{n}$. Moreover, since $\mathfrak{p}$ lies in the center of $\mathfrak{k}$, then the semisimple part of $\mathfrak{l}$ coincides with the semisimple part of $\mathfrak{k}$. Hence, L also acts transitively on $\mathfrak{n}$ and $\mathfrak{n}$ is an irreducible $Ad\,L$-module as well. Consequently, any G-invariant metric on M is adapted and moreover is binormal, by the irreducibility of $\mathfrak{p}$ and $\mathfrak{n}$. Furthermore, $C_{\mathfrak{p}}$ must be scalar on $\mathfrak{n}$, since $C_{\mathfrak{k}}$ and $C_{\mathfrak{l}}$ are scalar on $\mathfrak{n}$. Therefore, G-invariant Einstein metrics are given by positive solutions of (2.9) in Theorem 2.1. Since $\gamma = c_{\mathfrak{l},\mathfrak{p}} = 0$, (2.9) is just a degree-one equation whose solution is

$$X = \frac{1+2b}{1+2c_{\mathfrak{k},\mathfrak{n}}}, \tag{2.14}$$

where b is the eigenvalue of $C_{\mathfrak{p}}$ on $\mathfrak{n}$. Now we compute b, which is the eigenvalue of $C_{\mathfrak{p}}$ on $\mathfrak{n}$. Since $\mathfrak{g}$ is simple we have $tr(C_{\mathfrak{p}}) = dim\,\mathfrak{p} = 1$. Since $\mathfrak{p}$ lies in the center of $\mathfrak{k}$, $C_{\mathfrak{p}}$ vanishes on $\mathfrak{k}$ and thus $tr(C_{\mathfrak{p}}) = tr(C_{\mathfrak{p}}\mid_{\mathfrak{n}}) = b\dim\mathfrak{n} = bm$. Hence,

$$b = \frac{1}{m}.$$

By replacing b on (2.14) we obtain the desired expression for X.

□

Example 2.1. ***Circle Bundles over Compact Irreducible Hermitian Symmetric Spaces.*** *An application of Corollary 2.5 occurs when the base space is an irreducible symmetric space. So let us consider a fibration $F \to M \to N$ where F is isomorphic to the circle group and N is an isotropy irreducible symmetric space. Since F is the circle group, $\mathfrak{p}$ lies in the center of $\mathfrak{k}$. Hence, K has one-dimensional center, since for a compact irreducible symmetric space the center of K has at most dimension 1. Moreover, in this case N is a compact irreducible Hermitian symmetric space. In particular, L must coincide with the semisimple part of K. Compact irreducible Hermitian symmetric spaces G/K are classified (see e.g. [16]). All the possible G, K and L are listed in Table 2.1, together with the coefficient X of the, unique, Einstein adapted metric on G/L, as in Corollary 2.5.*

Finally, if F is not isotropy irreducible, under some hypothesis we can show the following:

Table 2.1: Circle Bundles Over irreducible Hermitian Symmetric Spaces.

G	K	L	X
$SU(n)$	$S(U(p)\times U(n-p))$	$SU(p)\times SU(n-p)$	$\frac{p(n-p)+1}{2p(n-p)}$
$SO(2n)$	$U(n)$	$SU(n)$	$\frac{n(n-1)+2}{2n(n-1)}$
$SO(n)$	$SO(2)\times SO(n-2)$	$SO(n-2)$	$\frac{n-1}{n-2}$
$Sp(n)$	$U(n)$	$SU(n)$	$\frac{n(n+1)+2}{2n(n+1)}$
E_6	$SO(10)\times U(1)$	$SO(10)$	$\frac{17}{32}$
E_7	$E_6\times U(1)$	E_6	$\frac{14}{27}$

Corollary 2.6. *Suppose F is not isotropy irreducible and that there exists a constant α such that*

$$Kill_{\mathfrak{l}}\,|_{\mathfrak{p}\times\mathfrak{p}}=\alpha Kill_{\mathfrak{k}}\,|_{\mathfrak{p}\times\mathfrak{p}}\,.$$

For $a=1,\ldots,s$, let γ_a be the constant determined by

$$Kill_{\mathfrak{k}}\,|_{\mathfrak{p}_a\times\mathfrak{p}_a}=\gamma_a Kill\,|_{\mathfrak{p}_a\times\mathfrak{p}_a}\,.$$

If for some $a,b=1,\ldots,s$, $\gamma_a\neq\gamma_b$, then there exists a binormal Einstein metric on M if and only if, for every $j=1,\ldots,n$,

$$c_{\mathfrak{l},j}=\left(1-\frac{1}{\sqrt{2\alpha+1}}\right)\left(c_{\mathfrak{k},j}+\frac{1}{2}\right) \tag{2.15}$$

and $C_{\mathfrak{p}}$ is scalar on each $\mathfrak{n}_j$, where $c_{\mathfrak{l},j}$ and $c_{\mathfrak{k},j}$ are the eigenvalues of $C_{\mathfrak{l}}$ and $C_{\mathfrak{k}}$, respectively, on $\mathfrak{n}_j$. In this case, there is a unique binormal Einstein metric, up to homothety, given by

$$B\,|_{\mathfrak{p}\times\mathfrak{p}}\oplus\frac{1}{\sqrt{2\alpha+1}}B\,|_{\mathfrak{n}\times\mathfrak{n}}\,.$$

Proof: If $Kill_{\mathfrak{l}}\,|_{\mathfrak{p}\times\mathfrak{p}}=\alpha Kill_{\mathfrak{k}}\,|_{\mathfrak{p}\times\mathfrak{p}}$, then

$$c_{\mathfrak{l},a}=\alpha\gamma_a,\ \text{for every } a=1,\ldots,s. \tag{2.16}$$

Therefore, for any $a,b=1,\ldots,s$, if $s>1$, $2\delta^{\mathfrak{l}}_{ab}+\delta^{\mathfrak{k}}_{ab}=(2\alpha+1)\delta^{\mathfrak{k}}_{ab}$ and, thus, Equation (2.8) in Theorem 2.1 becomes

$$(2\alpha+1)\delta^{\mathfrak{k}}_{ab}X^2=\delta^{\mathfrak{k}}_{ab}. \tag{2.17}$$

In particular, (2.16) implies that $c_{\mathfrak{l},a}=0$ if and only if $\gamma_a=0$ (thus if $\mathfrak{p}$ has submodules where L acts trivially, then $Kill_{\mathfrak{k}}$ vanish on those submodules and then they lie in the center of $\mathfrak{k}$. If K is semisimple, then the isotropy representation of K/L is faithful). The fact that the isotropy representation of K/L is not irreducible implies that $\mathfrak{p}$ decomposes as a direct sum $\mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_s$ with $s > 1$. For the indices for which $\gamma_a \neq \gamma_b$[1], we have $\delta^{\mathfrak{k}}_{ab} \neq 0$ and (2.17) implies that

$$X = \frac{1}{\sqrt{2\alpha + 1}}.$$

Hence, $X = \frac{1}{\sqrt{2\alpha+1}}$ must be a root of the polynomial in (2.9). By using the fact that $c_{\mathfrak{l},a} = \alpha\gamma_a$ and $b_j = c_{\mathfrak{k},j} - c_{\mathfrak{l},j}$, simple calculations show that

$$c_{\mathfrak{l},j} = \left(1 - \frac{1}{\sqrt{2\alpha + 1}}\right)\left(c_{\mathfrak{k},j} + \frac{1}{2}\right). \tag{2.18}$$

We observe that this condition implies (2.7) in Theorem 2.1, as we can see by the equalities below:

$$\delta^{\mathfrak{l}}_{ij} = c_{\mathfrak{l},i} - c_{\mathfrak{l},j} = \left(1 - \frac{1}{\sqrt{2\alpha + 1}}\right)(c_{\mathfrak{k},i} - c_{\mathfrak{k},j}) = (1 - X)\delta^{\mathfrak{k}}_{ij}.$$

Hence, there is a binormal Einstein metric if and only if (2.18) is satisfied and the operator $C_{\mathfrak{p}}$ is scalar on $\mathfrak{n}_j$, for every $j = 1, \ldots, n$. In this case, according also to Theorem 2.1 such metric is, up to homothety, given by $B \mid_{\mathfrak{p}\times\mathfrak{p}} \oplus \frac{1}{\sqrt{2\alpha+1}} B \mid_{\mathfrak{n}\times\mathfrak{n}}$.
□

Corollary 2.7. *Suppose F is not isotropy irreducible and that there exists a constant α such that*

$$Kill_{\mathfrak{l}} \mid_{\mathfrak{p}\times\mathfrak{p}} = \alpha Kill_{\mathfrak{k}} \mid_{\mathfrak{p}\times\mathfrak{p}} .$$

For $a = 1, \ldots, s$, let γ_a be the constant determined by

$$Kill_{\mathfrak{k}} \mid_{\mathfrak{p}_a\times\mathfrak{p}_a} = \gamma_a Kill \mid_{\mathfrak{p}_a\times\mathfrak{p}_a} .$$

If for some $a, b = 1, \ldots, s$, $\gamma_a \neq \gamma_b$ and there exists on M an Einstein binormal metric, then the number $\sqrt{2\alpha + 1}$ is a rational.

Proof: This follows from the fact that the eigenvalues of $C_{\mathfrak{k}}$ and $C_{\mathfrak{l}}$ on $\mathfrak{n}_j$ are rational numbers. Since $\mathfrak{k}$ is a compact algebra, the eigenvalue of its Casimir operator on the complex representation on $\mathfrak{n}_j^{\mathbb{C}}$ is given by

$$\frac{< \lambda_j, \lambda_j + 2\delta >}{2h^*(\mathfrak{g})} \in \mathbb{Q},$$

where λ_j is the highest weight for $\mathfrak{n}_j^{\mathbb{C}}$, 2δ is the sum of all positive roots of $\mathfrak{k}$ and $h^*(\mathfrak{g})$ is the dual Coxeter number of $\mathfrak{g}$ ([17], [35]). A similar formula holds for

[1] This condition implies that the representation of L on at least one of the $\mathfrak{p}_a$'s is faithful.

$C_{\mathfrak{l},j}$ and we conclude that $C_{\mathfrak{l},j}$ and $C_{\mathfrak{k},j}$ are rational numbers. If there exists a binormal Einstein metric on M, then $C_{\mathfrak{l},j}$ and $C_{\mathfrak{k},j}$ are related by formula (2.15) in Corollary 2.6. This implies that $\sqrt{2\alpha+1}$ is a rational number.
□

2.4 Riemannian Fibrations with Einstein Fiber and Einstein Base

In this section we investigate conditions for the existence of an Einstein adapted metric $g_M = (\lambda_1, \ldots, \lambda_s;\, \mu_1, \ldots, \mu_n)$ on M such that g_F or g_N are also Einstein.

Theorem 2.2. *Let $g_M = g_M(\lambda_1, \ldots, \lambda_s;\, \mu_1, \ldots, \mu_n)$ be an adapted metric on M. If g_M and g_N are both Einstein and $n > 1$, then*

$$\frac{\mu_j}{\mu_k} = \frac{r_j}{r_k} = \left(\frac{b^j}{b^k}\right)^{\frac{1}{2}}, \text{ for } j, k = 1, \ldots, n,$$

where b^j is the eigenvalue of the operator $\sum_{a=1}^{s} \lambda_a C_{\mathfrak{p}_a}$ on $\mathfrak{n}_j$, for $j = 1, \ldots, n$, and the r_j's are determined by $Ric^N = \oplus_{k=1}^{n} r_k B \mid_{\mathfrak{n}_k \times \mathfrak{n}_k}$ as in Lemma 1.9. Up to homothety, there exists at most one Einstein metric g_N on N such that the corresponding g_M on M is Einstein.

Proof: Let $g_M(\lambda_1, \ldots, \lambda_s;\, \mu_1, \ldots, \mu_n)$ be an adapted metric on M. From Corollary 1.4 we know that if g_M is Einstein, then there are constants b^j such that

$$\sum_{a=1}^{s} \lambda_a C_{\mathfrak{p}_a} \mid_{\mathfrak{n}_j} = b^j Id_{\mathfrak{n}_j}.$$

We recall from Lemma 1.9 that $Ric^N = \oplus_{k=1}^{n} r_k B \mid_{\mathfrak{n}_k \times \mathfrak{n}_k}$. Hence, if g_N is Einstein, then

$$\frac{r_1}{\mu_1} = \ldots = \frac{r_n}{\mu_n}. \tag{2.19}$$

From this equalities we obtain that

$$\frac{\mu_j}{\mu_k} = \frac{r_j}{r_k}, \text{ for } j, k = 1, \ldots, n. \tag{2.20}$$

From Proposition 1.2, for $X \in \mathfrak{n}_k$, the Ricci curvature of g_M is

$$Ric(X, X) = -\frac{1}{2\mu_k} \sum_{a=1}^{s} \lambda_a B(C_{\mathfrak{p}_a} X, X) + r_k B(X, X) = \big(-\frac{b^k}{2\mu_k} + r_k\big) B(X, X).$$

If g_M is Einstein, then from the expression above we obtain the following Equations

$$-\frac{b^k}{2\mu_k^2}+\frac{r_k}{\mu_k}=-\frac{b^j}{2\mu_j^2}+\frac{r_j}{\mu_j}. \tag{2.21}$$

The identities (2.19) and (2.21) imply that

$$\frac{b^k}{\mu_k^2}=\frac{b^j}{\mu_j^2}$$

and consequently, by using (2.20),

$$\left(\frac{r_j}{r_k}\right)^2=\left(\frac{\mu_j}{\mu_k}\right)^2=\frac{b^j}{b^k}.$$

Finally, we observe that although the fact that g_N is Einstein implies the equalities $\frac{\mu_j}{\mu_k}=\frac{r_j}{r_k}$, there might be more then one solution for the n-tuples $(\mu_1,\dots,\mu_n)$, up to scalar multiplication, since the r_j's in general depend on the μ_i's. This is obvious since clearly there might be many distinct Einstein metrics on N, up to homothety. This is explicit in the formula given in Lemma 1.9. However, as the eigenvalues b^j are independent of the constants $\mu_1,\dots,\mu_n$, the ratios $\frac{\mu_j}{\mu_k}=\left(\frac{b^j}{b^k}\right)^{\frac{1}{2}}$ imply that there is at most one possible choice for g_N, up to scalar multiplication.

□

Theorem 2.3. *Let $g_M(\lambda_1,\dots,\lambda_s;\,\mu_1,\dots,\mu_n)$ be an adapted metric on M. If g_M and g_F are both Einstein and $s>1$, then*

$$\frac{\lambda_a}{\lambda_b}=\frac{q_a}{q_b}=\sum_{j=1}^{n}\frac{C_{\mathfrak{n}_j,b}}{\mu_j^2}\Big/\sum_{j=1}^{n}\frac{C_{\mathfrak{n}_j,a}}{\mu_j^2},\ \textit{for } a,b=1,\dots,s,$$

where $c_{\mathfrak{n}_j,a}$ is such that $Kill(C_{\mathfrak{n}_j}\cdot,\cdot)\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}=c_{\mathfrak{n}_j,a}Kill\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}$, for $a=1,\dots,s$ and the q_a's are determined by $Ric^F=\oplus_{a=1}^{s}q_aB\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}$ as in Lemma 1.7. Up to scalar multiplication, there exists at most one Einstein metric g_F on F such that the corresponding metric g_M on M is Einstein.

Proof: The proof is similar to that of Theorem 2.2, by using Lemma 1.7 and Proposition 1.1.

□

Corollary 2.8. *Let $g_M=g_M(\lambda_1,\dots,\lambda_s;\,\mu_1,\dots,\mu_n)$ be an adapted metric on M. If g_M, g_N and g_F are Einstein, then*

$$\frac{r_j}{r_k}=\left(\frac{b^j}{b^k}\right)^{\frac{1}{2}},\ \textit{for } j,k=1,\dots,n,$$

and

$$\frac{q_a}{q_b} = \sum_{j=1}^{n} \frac{C_{\mathfrak{n}_j,b}}{b_j} \Big/ \sum_{j=1}^{n} \frac{C_{\mathfrak{n}_j,a}}{b_j}, \text{ for } a, b = 1, \dots, s,$$

where all the constants are as in Theorems 2.2 and 2.3.

Proof: Using Theorem 2.2, we write $\mu_j^2 = \frac{b^j}{b^1}\mu_1^2$. The second formula follows immediately from this and Theorem 2.3.

□

Theorem 2.4. *Let $g_M = g_M(\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_n)$ be an adapted metric on M. Suppose that g_M, g_N and g_F are Einstein and let E, E_F and E_N be the corresponding Einstein constants. If $E \neq E_N$, then*

$$\mu_j = \left(\frac{b^j}{2(E_N - E)}\right)^{\frac{1}{2}},$$

$$\lambda_a = 2\frac{E - E_F}{E_N - E}\left(\frac{C_{\mathfrak{n}_j,a}}{b^j}\right)^{-1}.$$

where b^j is the eigenvalue of the operator $\sum_{a=1}^{s} \lambda_a C_{\mathfrak{p}_a}$ on $\mathfrak{n}_j$ and $c_{\mathfrak{n}_j,a}$ is such that $Kill(C_{\mathfrak{n}_j}\cdot, \cdot) \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} = c_{\mathfrak{n}_j,a} Kill \mid_{\mathfrak{p}_a \times \mathfrak{p}_a}$.

Proof: Let $g_M = (\lambda_1, \dots, \lambda_s; \mu_1, \dots, \mu_n)$ be an adapted metric on M. If g_M, g_N and g_F are all Einstein, from Propositions 1.1 and 1.2 we get

$$-\frac{1}{2\mu_j} b^j + \mu_j E_N = \mu_j E,$$

from which, if $E_N \neq E$, we deduce

$$\mu_j^2 = \frac{b^j}{2(E_N - E)} \tag{2.22}$$

and

$$\lambda_a E_F + \frac{\lambda_a^2}{4}\frac{C_{\mathfrak{n}_j,a}}{\mu_j^2} = \lambda_a E.$$

From this we get

$$\lambda_a \frac{C_{\mathfrak{n}_j,a}}{\mu_j^2} = 4(E - E_F). \tag{2.23}$$

We obtain the required formula by replacing (2.22) in the equation above.

□

2.5 Riemannian Fibrations with Symmetric Fiber

In this section we consider a fibration $F \to M \to N$ such that F is a symmetric space and N is isotropy irreducible. We specify the Ricci curvature of an adapted metric on M and obtain the Einstein equations in some particular cases.

If $F = K/L$ is a symmetric space, then we consider its DeRham decomposition

$$K/L = K_0/L_0 \times K_1/L_1 \times \ldots \times K_s/L_s, \tag{2.24}$$

where K_0 is the center of K and, for $a = 1, \ldots, s$, K_a is simple. By $\mathfrak{k}_a$ and $\mathfrak{l}_a$ we denote the Lie algebras of K_a and L_a, respectively. In particular, for $a = 1, \ldots, s$, K_a/L_a is an irreducible symmetric space. Thus $\mathfrak{p}_a$ may be chosen as a symmetric reductive complement of $\mathfrak{l}_a$ in $\mathfrak{k}_a$. Since $\mathfrak{k}_a$ is simple, the Casimir operator of $\mathfrak{k}$ is scalar on $\mathfrak{k}_a$. Hence, in the equality $Kill_{\mathfrak{k}} |_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a Kill |_{\mathfrak{p}_a \times \mathfrak{p}_a}$, the constant γ_a is simply the eigenvalue of the Casimir operator of $\mathfrak{k}$ on $\mathfrak{k}_a$, because $Kill_{\mathfrak{k}} = Kill(C_{\mathfrak{k}} \cdot, \cdot)$. For $a = 0$ this is still true with $\gamma_0 = 0$.

Proposition 2.4. *Suppose that F is a symmetric space and N is isotropy irreducible and let $g_M = (\lambda_0, \ldots, \lambda_s; \mu)$ be an adapted metric on M. The Ricci curvature of g_M is as follows:*
(i) For every $X \in \mathfrak{p}_a$, $a = 0, \ldots, s$,

$$Ric(X,X) = \left(\frac{\gamma_a}{2} + \frac{\lambda_a^2}{4\mu^2}(1-\gamma_a) \right) B(X,X),$$

where γ_a is the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{k}_a$;
(ii) For every $X \in \mathfrak{n}$,

$$Ric(X,X) = -\frac{1}{2} \sum_{a=1}^{s} \lambda_a B(C_{\mathfrak{p}_a} X, X) + rB(X,X),$$

where $r = \frac{1}{2}\big(\frac{1}{2} + c_{\mathfrak{k},\mathfrak{n}}\big)$ and $c_{\mathfrak{k},\mathfrak{n}}$ is the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{n}$;
(iii) $Ric(\mathfrak{p}, \mathfrak{n}) = 0$;
(iv) For every $a, b = 0, \ldots, s$ such that $a \neq b$, $Ric(\mathfrak{p}_a, \mathfrak{p}_b) = 0$ and for every $i, j = 1, \ldots, n$ such that $i \neq j$, $Ric(\mathfrak{n}_i, \mathfrak{n}_j) = 0$.

Proof: Since N is isotropy irreducible the expressions for the Ricci curvature of g_M are given by Proposition 2.3. In particular, for $X \in \mathfrak{p}_a$,

$$Ric(X,X) = \left(q_a + \frac{\lambda_a^2}{4\mu^2}(1-\gamma_a) \right) B(X,X).$$

If we consider the DeRham decomposition of F as in (2.24), $[\mathfrak{p}_a, \mathfrak{p}_b] = 0$, for every $a \neq b$. Hence, from Proposition 2.2, we have $q_a = \frac{1}{2}\left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a}\right)$. Since F is a symmetric space, then $Kill_{\mathfrak{l}}\mid_{\mathfrak{p}\times\mathfrak{p}}=\frac{1}{2}Kill_{\mathfrak{k}}\mid_{\mathfrak{p}\times\mathfrak{p}}$ and thus $C_{\mathfrak{l}}\mid_{\mathfrak{p}\times\mathfrak{p}}=\frac{1}{2}C_{\mathfrak{k}}\mid_{\mathfrak{p}\times\mathfrak{p}}$. Hence, $c_{\mathfrak{l},a}=\frac{\gamma_a}{2}$. Therefore $q_a=\dfrac{\gamma_a}{2}$.

Since N is irreducible, by using Proposition 2.3, for $X\in\mathfrak{n}$, we write

$$Ric(X,X)=-\frac{1}{2}\sum_{a=1}^{s}\frac{\lambda_a}{\mu}B(C_{\mathfrak{p}_a}X,X)+rB(X,X),$$

where

$$r=\frac{1}{2}\left(\frac{1}{2}+c_{\mathfrak{k},\mathfrak{n}}\right)$$

and $c_{\mathfrak{k},\mathfrak{n}}$ is the eigenvalue of the Casimir operator $C_{\mathfrak{k}}$ on $\mathfrak{n}$.

(iii) and (iv) follow directly from Proposition 2.3 as well.

□

Theorem 2.5. *Suppose that F is a symmetric space and N is an isotropy irreducible space. Moreover, suppose that $C_{\mathfrak{p}_a}\mid_{\mathfrak{n}}=b_aId_{\mathfrak{n}}$, for some constants b_a, for every $a=1,\dots,s$. There exists on M an Einstein adapted metric if and only if there are positive solutions of the following system of s algebraic equations in the unknowns $X_1,\dots,X_s$:*[2]

$$2\gamma_1X_1^2X_a+(1-\gamma_1)X_a-2\gamma_aX_1X_a^2-(1-\gamma_a)X_1=0,\ a=2,\dots,s$$

$$2\sum_{a=1}^{s}b_aX_1\dots\widehat{X_a}\dots X_s-4rX_1\dots X_s+2\gamma_1X_1^2X_2\dots X_s+(1-\gamma_1)X_2\dots X_s=0,$$

where γ_a is the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{p}_a$, $r=\frac{1}{2}\left(\frac{1}{2}+c_{\mathfrak{k},\mathfrak{n}}\right)$ and $c_{\mathfrak{k},\mathfrak{n}}$ is the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{n}$. To each s-tuple $(X_1,\dots,X_s)$ corresponds a family of Einstein adapted metrics on M given, up to homothety, by

$$g_M=\oplus_{a=1}^{s}\frac{1}{X_a}B\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}\oplus B\mid_{\mathfrak{n}\times\mathfrak{n}}.$$

Proof: Let $g_M=(\lambda_1,\dots,\lambda_s;\mu)$ be an adapted metric on M. First we observe that the hypothesis $C_{\mathfrak{p}_a}\mid_{\mathfrak{n}}=b_aId_{\mathfrak{n}}$, for every $a=1,\dots,s$, implies that $\sum_{a=1}^{s}\lambda_aC_{\mathfrak{p}_a}$ is scalar for any choice of λ_a's. Hence, the necessary condition for the existence of an Einstein adapted metric on M given by Corollary 1.4 is satisfied. Moreover, (iii) and (iv) of Proposition 2.4 imply that for g_M to be Einstein, it suffices to analyze the equations

$$Ric\mid_{\mathfrak{p}_a\times\mathfrak{p}_a}=\lambda_aEB\mid_{\mathfrak{p}_a\times\mathfrak{p}_a},\ a=1,\dots,s \tag{2.25}$$

$$Ric\mid_{\mathfrak{n}\times\mathfrak{n}}=\mu EB\mid_{\mathfrak{n}\times\mathfrak{n}}, \tag{2.26}$$

[2] $\widehat{X_a}$ means that X_a does not occur in the product.

where E is the Einstein constant of g_M.
We introduce the unknowns

$$X_a = \frac{\mu}{\lambda_a},\ a = 1, \dots, s.$$

By using $C_{\mathfrak{p}_a} |_{\mathfrak{n}} = b_a Id_{\mathfrak{n}}$ and the X_a's, Equation 2.26 may be rewritten as

$$-\sum_{a=1}^{s} \frac{b_a}{2X_a} + r = \mu E. \tag{2.27}$$

Also, by using Proposition 2.4 and the X_a's, Equation 2.25 may be rewritten as

$$\frac{\gamma_a}{2} + \frac{1-\gamma_a}{4X_a^2} = \lambda_a E. \tag{2.28}$$

By multiplying (2.28) by X_a we get

$$\frac{2\gamma_a X_a^2 + 1 - \gamma_a}{4X_a} = \mu E. \tag{2.29}$$

Therefore, the Einstein Equations are just

$$\frac{2\gamma_a X_a^2 + 1 - \gamma_a}{4X_a} = \frac{2\gamma_1 X_1^2 + 1 - \gamma_1}{4X_1},\ a = 1, \dots, s \tag{2.30}$$

$$-\sum_{a=1}^{s} \frac{b_a}{2X_a} + r = \frac{2\gamma_1 X_1^2 + 1 - \gamma_1}{4X_1}. \tag{2.31}$$

We obtain the equations stated in the theorem simply by rearranging (2.30) and (2.31). We recall that since N is irreducible, we have $r = \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{k},\mathfrak{n}}\right)$, as in Proposition 2.4, and thus r does not depend on μ. So $X_1, \dots, X_s$ are actually the only unknowns of the system above.
□

Corollary 2.9. *Suppose that F and N are irreducible symmetric spaces and $\dim F > 1$. There exists on M an Einstein adapted metric if and only if $C_{\mathfrak{p}}$ is scalar on $\mathfrak{n}$ and $\triangle' \geq 0$, where*

$$\triangle' = 1 - 2\gamma(1 - \gamma + 2b),$$

γ is the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{p}$ and b is the eigenvalue of $C_{\mathfrak{p}}$ on $\mathfrak{n}$. If these two conditions are satisfied, then Einstein adapted metrics are homothetic to $g_M = B |_{\mathfrak{p}\times\mathfrak{p}} \oplus XB |_{\mathfrak{n}\times\mathfrak{n}}$, where

$$X = \frac{1 \pm \sqrt{\triangle'}}{2\gamma}.$$

Proof: It follows from Corollary 2.4 and from the fact that, since F and N are irreducible symmetric spaces, $c_{\mathfrak{k},\mathfrak{n}} = \dfrac{1}{2}$ and $c_{\mathfrak{l},\mathfrak{p}} = \dfrac{\gamma}{2}$.
□

Corollary 2.10. *Suppose that F is a symmetric space and N is isotropy irreducible.*
(i) If $C_{\mathfrak{p}}$ is not scalar on $\mathfrak{n}$ or $C_{\mathfrak{k}}$ is not scalar on $\mathfrak{p}$, then there is no binormal Einstein metric on M.
(ii) Suppose that $C_{\mathfrak{p}}$ is scalar on $\mathfrak{n}$ and $C_{\mathfrak{k}}$ is scalar on $\mathfrak{p}$, and write $C_{\mathfrak{p}} \mid_{\mathfrak{n}} = bId_{\mathfrak{n}}$ and $C_{\mathfrak{k}} \mid_{\mathfrak{p}} = \gamma Id_{\mathfrak{p}}$. There is an one-to-one correspondence between binormal Einstein metrics on M and positive roots of the polynomial

$$2\gamma X^2 - (1 + 2c_{\mathfrak{k},\mathfrak{n}})\, X + (1 - \gamma + 2b) = 0. \tag{2.32}$$

for every $a, b = 1, \ldots, s$ and $i, j = 1, \ldots, n$, where $c_{\mathfrak{k},\mathfrak{n}}$ is the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{n}$. If such a positive solution X exists, then binormal Einstein metrics are, up to homothety, given by

$$<,>= B \mid_{\mathfrak{p}\times\mathfrak{p}} \oplus XB \mid_{\mathfrak{n}\times\mathfrak{n}} .$$

Proof: If F is a symmetric space, then $Kill_{\mathfrak{l}} \mid_{\mathfrak{p}\times\mathfrak{p}} = \alpha Kill_{\mathfrak{k}} \mid_{\mathfrak{p}\times\mathfrak{p}}$, with $\alpha = \frac{1}{2}$. The number $\sqrt{2\alpha + 1} = \sqrt{2}$ is not a rational. Hence, Corollary 2.7 implies that if there exists a binormal Einstein metric on M, then $\gamma_1 = \ldots = \gamma_s = \gamma$ for some constant γ, i.e., the Casimir operator of $\mathfrak{k}$ is scalar on $\mathfrak{p}$.
Moreover, we know from Theorem 2.1 that the condition that $C_{\mathfrak{p}}$ is scalar on $\mathfrak{n}$ is also a necessary condition for the existence of a binormal Einstein metric.
The polynomial (2.32) is just (2.9) from Theorem 2.1, for $c_{\mathfrak{l},\mathfrak{p}} = \frac{\gamma}{2}$ and $n = 1$. Also, the condition (2.8) from Theorem 2.1 is satisfied since for $\gamma_1 = \ldots = \gamma_s$, we have $\delta^{\mathfrak{k}}_{ab} = \delta^{\mathfrak{l}}_{ab} = 0$.
□

Corollary 2.11. *Suppose that F is a symmetric space. If there exists on M a binormal Einstein metric g_M, then g_F is Einstein. The converse holds if $C_{\mathfrak{k}}$ is scalar on $\mathfrak{p}$.*

Proof: If F is irreducible, then any metric on F is Einstein. So let us suppose that F is a reducible symmetric space. Then, by Corollary 2.10, the existence of a binormal Einstein metric g_M on M implies that $\gamma_1 = \ldots = \gamma_s = \gamma$ for some γ. From Proposition 2.1, we have $Ric^F \mid_{\mathfrak{p}_a\times\mathfrak{p}_a} = q_a B(X, X)$, where $q_a = \frac{1}{2}\left(\frac{\gamma_a}{2} + c_{\mathfrak{l},a}\right) = \frac{\gamma_a}{2} = \frac{\gamma}{2}$, for every $a = 1, \ldots, s$. Therefore, g_F is Einstein with Einstein constant $E_F = \frac{\gamma}{2\lambda}$.

Conversely, let $g_M = (\lambda_1, \ldots, \lambda_s, \mu)$ be any Einstein adapted metric on M. If $C_{\mathfrak{k}}$ is scalar on $\mathfrak{p}$, then $\gamma_1 = \ldots = \gamma_s$. Hence, if g_F is Einstein, we have from Theorem 2.3 that

$$\frac{\lambda_a}{\lambda_b} = \frac{q_a}{q_b} = \frac{\gamma_a/2}{\gamma_b/2} = 1$$

and g_M is binormal.

□

Hence, binormal Einstein metrics are such that the restriction to the fiber is Einstein. As the next two results show, there might exists other Einstein adapted metrics satisfying this property.

Corollary 2.12. *Suppose that F is a symmetric space and N is isotropy irreducible. Let γ_a be the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{p}_a$, $a = 1, \ldots, s$, and g_M an adapted metric on M. If g_M and g_F are both Einstein, then*

$$\gamma_a = \gamma_b \text{ or } \gamma_a = 1 - \gamma_b, \; a, b = 1, \ldots, s.$$

Proof: If F is a symmetric space, we have $q_a = \frac{\gamma_a}{2}$, for every $a = 1, \ldots, s$. On the other hand, if N is isotropy irreducible, then $\mathfrak{n}$ is an irreducible $Ad\,K$-module, and since $C_{\mathfrak{n},a} = 1 - \gamma_a$, the identity in Theorem 2.3 becomes

$$\frac{\gamma_a/2}{\gamma_b/2} = \frac{1-\gamma_b}{\mu^2}\Big/\frac{1-\gamma_a}{\mu^2}.$$

Hence, we obtain the equation

$$\gamma_a(1-\gamma_a) = \gamma_b(1-\gamma_b), \text{ for } a, b = 1, \ldots, s,$$

whose solutions are $\gamma_a = \gamma_b$ or $\gamma_a = 1 - \gamma_b$.

□

Corollary 2.13. *Suppose that F is a symmetric space such that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are non-abelian, and N is isotropy irreducible. Suppose that $C_{\mathfrak{p}_a}\mid_{\mathfrak{n}} = b_a Id_{\mathfrak{n}}$, for some constants b_a, $a = 1, 2$. Let γ_a be the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{p}_a$, $a = 1, 2$, and $c_{\mathfrak{k},\mathfrak{n}}$ be the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{n}$.*
If there exists on M an Einstein adapted metric g_M such that g_F is also Einstein, then one of the following cases holds:
(i) $\gamma_2 = \gamma_1$ and $\triangle \geq 0$, where

$$\triangle = (1 + c_{\mathfrak{k},\mathfrak{n}})^2 - 8\gamma_1(1 - \gamma_1 + 2b).$$

If these two conditions are satisfied the metric, then g_M is the binormal metric given, up to homothety, by

$$g_M = B\mid_{\mathfrak{p}\times\mathfrak{p}} \oplus XB\mid_{\mathfrak{n}\times\mathfrak{n}},\ where\, X = \frac{1 + c_{\mathfrak{k},\mathfrak{n}} \pm \sqrt{\triangle}}{2\gamma_1}.$$

(ii) $\gamma_2 = 1 - \gamma_1$ *and* $D(\gamma_1) \geq 0$, *where*

$$D(\gamma_1) = 4r^2 - 4b_1\gamma_1 - 4b_2(1-\gamma_1) - 2\gamma_1(1-\gamma_1)$$

and $r = \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{k},\mathfrak{n}}\right)$. *If these two conditions are satisfied the metric g_M is given, up to homothety, by*

$$g_M = \frac{1}{X_1}B\mid_{\mathfrak{p}_1\times\mathfrak{p}_1} \oplus \frac{1}{X_2}B\mid_{\mathfrak{p}_2\times\mathfrak{p}_2} \oplus B\mid_{\mathfrak{n}\times\mathfrak{n}},$$

where

$$X_2 = \frac{\gamma_1 X_1}{1-\gamma_1}\ \ and\ X_1 = \frac{2r \pm \sqrt{D(\gamma_1)}}{2\gamma_1}.$$

Proof: First we observe that the hypothesis that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are non-abelian implies that $\gamma_1, \gamma_2 \neq 0$. Let $g_M = g_M(\lambda_1, \ldots, \lambda_s, \mu)$ be an Einstein adapted metric on M such that g_F is also Einstein. Corollary 2.12, implies that either $\gamma_2 = \gamma_1$ or $\gamma_2 = 1 - \gamma_1$.

In the case $\gamma_2 = \gamma_1$, the statement follows from Corollaries 2.10 and 2.11. In the case $\gamma_2 = 1 - \gamma_1$, we obtain from Theorem 2.3, that

$$\frac{\lambda_1}{\lambda_2} = \frac{\gamma_1/2}{\gamma_2/2} = \frac{\gamma_1}{1-\gamma_1}. \tag{2.33}$$

On the other hand, according to Theorem 2.5, an adapted Einstein metric on M corresponds to positive solutions of the equations

$$2\gamma_1 X_1^2 X_2 + (1-\gamma_1)X_2 - 2\gamma_2 X_1 X_2^2 - (1-\gamma_2)X_1 = 0 \tag{2.34}$$

$$2b_1X_2 + 2b_2X_1 - 4rX_1X_2 + 2\gamma_1X_1^2X_2 + (1-\gamma_1)X_2 = 0, \tag{2.35}$$

where $X_a = \frac{\mu}{\lambda_a}$. By using the identity (2.33), we solve the system of equations (2.34) and (2.35) for $X_2 = \frac{\gamma_1}{1-\gamma_1}X_1$ and $\gamma_2 = 1-\gamma_1$, in order to obtain the solutions stated in (ii).

□

The following two Corollaries classify all the Einstein adapted metrics in the cases when $\gamma_2 = \gamma_1$ or $\gamma_2 = 1 - \gamma_1$. These results follow immediately from Corollary 2.13 and from solving the equations (2.34) and (2.35) for $\gamma_2 = \gamma_1$ or $\gamma_2 = 1 - \gamma_1$, respectively.

Corollary 2.14. *Suppose that F is a symmetric space such that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are non-abelian, and N is isotropy irreducible. Suppose that $C_{\mathfrak{p}_a} |_{\mathfrak{n}} = b_a Id_{\mathfrak{n}}$, for some constants b_a, $a = 1, 2$, and let γ_a be the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{p}_a$, $a = 1, 2$, and $c_{\mathfrak{k},\mathfrak{n}}$ the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{n}$.*
Suppose that $\gamma_2 = \gamma_1$, i.e., $C_{\mathfrak{k}}$ is scalar on $\mathfrak{p}$. If there exists on M an Einstein adapted metric g_M, then one of the following two cases holds:
(i) g_F is also Einstein and g_M is a binormal metric given by Corollary 2.13 (i).
(ii) $D(\gamma_1) \geq 0$, where

$$D(\gamma_1) = 4r^2(1-\gamma_1) - 2\gamma_1(2b_2 + 1 - \gamma_1)(2b_1 + 1 - \gamma_1)$$

and $r = \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{k},\mathfrak{n}}\right)$. The metric g_M is given, up to homothety, by

$$g_M = \frac{1}{X_1} B |_{\mathfrak{p}_1 \times \mathfrak{p}_1} \oplus \frac{1}{X_2} B |_{\mathfrak{p}_2 \times \mathfrak{p}_2} \oplus B |_{\mathfrak{n} \times \mathfrak{n}},$$

where

$$X_2 = \frac{1-\gamma_1}{2\gamma_1 X_1} \text{ and } X_1 = \frac{2r(1-\gamma_1) \pm \sqrt{(1-\gamma_1)D(\gamma_1)}}{2\gamma_1(2b_2 + 1 - \gamma_1)}.$$

In this second case, g_F is not Einstein and g_M is not binormal.

Corollary 2.15. *Suppose that F is a symmetric space such that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are non-abelian, and N is isotropy irreducible. Suppose that $C_{\mathfrak{p}_a} |_{\mathfrak{n}} = b_a Id_{\mathfrak{n}}$, for some constants b_a, $a = 1, 2$, and let γ_a be the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{p}_a$, $a = 1, 2$, and $c_{\mathfrak{k},\mathfrak{n}}$ the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{n}$.*
Suppose that $\gamma_2 = 1 - \gamma_1$. If there exists on M an Einstein adapted metric g_M, then one of the following two cases holds:
(i) g_F is also Einstein and g_M is the metric given by Corollary 2.13 (ii).
(ii) $D(\gamma_1) \geq 0$, where

$$D(\gamma_1) = 4r^2 - 2(2b_2 + \gamma_1)(2b_1 + 1 - \gamma_1)$$

and $r = \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{k},\mathfrak{n}}\right)$. The metric g_M is given, up to homothety, by

$$g_M = \frac{1}{X_1} B |_{\mathfrak{p}_1 \times \mathfrak{p}_1} \oplus \frac{1}{X_2} B |_{\mathfrak{p}_2 \times \mathfrak{p}_2} \oplus B |_{\mathfrak{n} \times \mathfrak{n}},$$

where

$$X_2 = \frac{1}{2X_1} \text{ and } X_1 = \frac{2r \pm \sqrt{D(\gamma_1)}}{2(2b_2 + \gamma_1)}.$$

g_M is never binormal and in the second case g_F is not Einstein.

CHAPTER 3

As in the previous chapters, we consider a homogeneous fibration $F \to M \to N$, for $M = G/L$, $N = G/K$ and $F = K/L$, where G is a compact connected semisimple Lie group and $L \subsetneq K \subsetneq G$ connected closed non-trivial subgroups. In this chapter we suppose that both the fiber F and the base space N are symmetric spaces of maximal rank and, moreover, N is isotropy irreducible. The triple formed by the Lie algebras of G, K, L, denoted by $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$, shall be called a bisymmetric triple of maximal rank. We classify all the bisymmetric triple of maximal rank when $\mathfrak{g}$ is simple and obtain formulas to compute the eigenvalues which are necessary to decide about the existence of Einstein adapted metrics. For each triple, we present the eigenvalues of the Casimir operators of the irreducible L-invariant subspaces of the fiber on the horizontal direction and the eigenvalues of the Casimir operator of $\mathfrak{k}$ on the vertical direction. The computations for these eigenvalues are in Appendix A as well as a description of the isotropy representation in terms of subset of roots for each triple. Finally, we study the existence of adapted Einstein metrics by using the results in previous chapters. Tables are presented in the end of this chapter. We use the notation used in previous chapters unless stated otherwise.

3.1 Introduction

For all the definitions and properties concerning the roots system of a Lie algebra please see [16] or [34]. Let G be a compact connected semisimple Lie group and $L \subsetneq K \subsetneq G$ connected closed non-trivial subgroups such that $N = G/K$ is isotropy irreducible. As in section 1.2 of Chapter 1, $\mathfrak{n}$ and $\mathfrak{p}$ denote the reductive complements of $\mathfrak{k}$ in $\mathfrak{g}$ and of $\mathfrak{l}$ in $\mathfrak{k}$, respectively. The subspace $\mathfrak{n}$ is irreducible as an $Ad\,K$-module and $\mathfrak{p}$ may decompose into the direct sum $\mathfrak{p} = \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_s$ of irreducible $Ad\,L$-modules. We suppose that M has simple spectrum, i.e., $\mathfrak{n}$ do not contain any $Ad\,L$-submodule equivalent to any of the $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are pairwise inequivalent.

Initially, we only suppose that L is a subgroup of maximal rank in G. We choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$ such that $\mathfrak{h} \subset \mathfrak{l}^{\mathbb{C}}$. Let $\mathcal{R}$ be a system of nonzero roots for $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}$. As usual we have a decomposition of $\mathfrak{g}^{\mathbb{C}}$ into root subspaces

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} \oplus (\oplus_{\alpha \in \mathcal{R}} \mathfrak{g}^{\alpha}),$$

where $\mathfrak{g}^{\alpha} = \{u \in \mathfrak{g}^{\mathbb{C}} : ad\, h(u) = \alpha u, \forall h \in \mathfrak{h}\}$.
We have $Kill(\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}) \neq 0$ if and only if $\alpha + \beta = 0$ and thus, for every $\alpha \in \mathcal{R}$, we can take $E_{\alpha} \in \mathfrak{g}^{\alpha}$ such that $Kill(E_{\alpha}, E_{-\alpha}) = 1$. Since $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}] \subset \mathfrak{h}$, each pair $E_{\alpha}, E_{-\alpha}$ determines an element H_{α} in the Cartan subalgebra $\mathfrak{h}$ given by $H_{\alpha} = [E_{\alpha}, E_{-\alpha}]$. The vectors H_{α} are such that $Kill(H_{\alpha}, h) = \alpha(h)$, for every $h \in \mathfrak{h}$. In particular, the length $|\alpha|$ of a root $\alpha \in \mathcal{R}$ in $\mathfrak{g}$ is defined by

$$|\alpha|^2 = \alpha(H_{\alpha}) = Kill(H_{\alpha}, H_{\alpha}). \tag{3.1}$$

For every $\alpha, \beta \in \mathcal{R}$ such that $\alpha + \beta \in \mathcal{R}$, since $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}] \subset \mathfrak{g}^{\alpha+\beta}$, we define numbers $N_{\alpha,\beta} \in \mathbb{C}$ by

$$[E_{\alpha}, E_{\beta}] = N_{\alpha,\beta} E_{\alpha+\beta}, \tag{3.2}$$

called the *structure constants*. The $N_{\alpha,\beta}$'s satisfy the following properties:

$$N_{\alpha,\beta} = -N_{\beta,\alpha} \tag{3.3}$$

$$N_{-\alpha,\beta+\alpha} = N_{-\beta,-\alpha} = N_{\alpha,\beta}, \tag{3.4}$$

for every $\alpha, \beta \in \mathcal{R}$ such that $\alpha + \beta \in \mathcal{R}$.
A basis $\{E_{\alpha}\}_{\alpha \in \mathcal{R}}$ of $\oplus_{\alpha \in R} \mathfrak{g}^{\alpha}$ formed by elements chosen as above is called a *standard normalized basis* and that is what we will use throughout. By using such a basis we construct the elements

$$X_{\alpha} = \frac{E_{\alpha} - E_{-\alpha}}{\sqrt{2}} \text{ and } Y_{\alpha} = \frac{i(E_{\alpha} + E_{-\alpha})}{\sqrt{2}}. \tag{3.5}$$

The vectors X_{α} and Y_{α} are unit vectors with respect to B. Together with the maximal toral subalgebra $i\mathfrak{h}_{\mathbb{R}}$, X_{α} and Y_{α} generate a compact real form for $\mathfrak{g}^{\mathbb{C}}$ which we identify with $\mathfrak{g}$ (see e.g. [16], ch.III).

Since $L \subset K$, $\mathfrak{h}$ is also a Cartan subalgebra for $\mathfrak{k}^{\mathbb{C}}$. We define the following subsets of roots

$$\mathcal{R}_{\mathfrak{l}} = \{\alpha \in \mathcal{R} : E_\alpha \in \mathfrak{l}^{\mathbb{C}}\} \tag{3.6}$$

$$\mathcal{R}_{\mathfrak{k}} = \{\alpha \in \mathcal{R} : E_\alpha \in \mathfrak{k}^{\mathbb{C}}\} \tag{3.7}$$

$$\mathcal{R}_{\mathfrak{n}} = \mathcal{R} - \mathcal{R}_{\mathfrak{k}} = \{\alpha \in \mathcal{R} : E_\alpha \in \mathfrak{n}^{\mathbb{C}}\} \tag{3.8}$$

$$\mathcal{R}_{\mathfrak{p}} = \mathcal{R}_{\mathfrak{k}} - \mathcal{R}_{\mathfrak{l}} = \{\alpha \in \mathcal{R} : E_\alpha \in \mathfrak{p}^{\mathbb{C}}\} \tag{3.9}$$

We may also consider the subsets of roots

$$\mathcal{R}_{\mathfrak{p}_a} = \{\alpha \in \mathcal{R} : E_\alpha \in \mathfrak{p}_a^{\mathbb{C}}\},\ a = 1, \ldots, s, \tag{3.10}$$

and, since $\mathfrak{l}$ has maximal rank and $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$, $\mathfrak{p}_a =< X_\alpha, Y_\alpha : \alpha \in \mathcal{R}^+_{\mathfrak{p}_a} >$. Since $Kill(E_\alpha, E_{-\alpha}) = 1$, the bases $\{E_\alpha\}_{\alpha \in \mathcal{R}_{\mathfrak{p}_a}}$ and $\{E_{-\alpha}\}_{\alpha \in \mathcal{R}_{\mathfrak{p}_a}}$ of $\mathfrak{p}_a^{\mathbb{C}}$ are dual with respect to $Kill$. Moreover, $\{X_\alpha, Y_\alpha\}_{\alpha \in \mathcal{R}^+_{\mathfrak{p}_a}}$ is an orthonormal basis for $\mathfrak{p}_a$ with respect to B. Consequently, the Casimir operators of $\mathfrak{p}_a^{\mathbb{C}}$ and of $\mathfrak{p}_a$ are

$$C_{\mathfrak{p}_a^{\mathbb{C}}} = \sum_{\alpha \in \mathcal{R}_{\mathfrak{p}_a}} ad_{E_\alpha} ad_{E_{-\alpha}} \tag{3.11}$$

$$C_{\mathfrak{p}_a} = - \sum_{\alpha \in \mathcal{R}^+_{\mathfrak{p}_a}} \left(ad^2_{X_\alpha} + ad^2_{Y_\alpha}\right). \tag{3.12}$$

Since $\mathfrak{k}$ has maximal rank $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$, we have $\mathfrak{n}^{\mathbb{C}} =< E_\alpha : \alpha \in \mathcal{R}_{\mathfrak{n}} >$ and $\mathfrak{n} =< X_\alpha, Y_\alpha : \alpha \in \mathcal{R}^+_{\mathfrak{n}} >$. The subspace $\mathfrak{n}$ is by hypothesis irreducible as an $Ad\,K$-submodule. If $\mathfrak{n} = \oplus_j \mathfrak{n}^j$ is a decomposition of $\mathfrak{n}$ into irreducible $Ad\,L$-modules, we write

$$\mathcal{R}_{\mathfrak{n}^j} = \{\phi \in \mathcal{R} : E_\phi \in (\mathfrak{n}^j)^{\mathbb{C}}\} \text{ and } \mathfrak{n}^j = \{X_\phi, Y_\phi : \phi \in \mathcal{R}^+_{\mathfrak{n}^j}\}. \tag{3.13}$$

We recall that one of the conditions for existence of an Einstein adapted metric on M, given in Corollary 1.4, is that there are $\lambda_1, \ldots, \lambda_s > 0$ such that the operator $\sum_{a=1}^s \lambda_a C_{\mathfrak{p}_a}$ is scalar on $\mathfrak{n}$. The Casimir operator $C_{\mathfrak{p}_a}$ is necessarily scalar on the irreducible $Ad\,L$-submodules $\mathfrak{n}^j$. Since $B(X_\phi, X_\phi) = 1$, the eigenvalue of $C_{\mathfrak{p}_a}$ on $\mathfrak{n}^j$ is given by $B(C_{\mathfrak{p}_a} X_\phi, X_\phi)$, for any $\phi \in \mathcal{R}^+_{\mathfrak{n}^j}$. We shall write b_a^ϕ for this eigenvalue, i.e.,

$$b_a^\phi = B(C_{\mathfrak{p}_a} X_\phi, X_{-\phi}) \tag{3.14}$$

$$C_{\mathfrak{p}_a} \mid_{\mathfrak{n}^j} = b_a^\phi Id_{\mathfrak{n}^j},\ \forall\, \phi \in \mathcal{R}_{\mathfrak{n}^j}. \tag{3.15}$$

Furthermore, the eigenvalue of $C_{\mathfrak{p}_a}$ on $\mathfrak{n}^j$, b_a^ϕ, must coincide with the eigenvalue of $C_{\mathfrak{p}_a^{\mathbb{C}}}$ on $(\mathfrak{n}^j)^{\mathbb{C}}$. Hence, we also have

$$b_a^\phi = Kill(C_{\mathfrak{p}_a^{\mathbb{C}}} E_\phi, E_{-\phi}). \tag{3.16}$$

Remark 3.1. *We observe that in previous Sections the notation $\mathfrak{n}_j$ was used to denote $Ad\,K$-irreducible submodules of $\mathfrak{n}$, while in here we use the similar notation $\mathfrak{n}^j$ to denote $Ad\,L$-irreducible submodules, whereas $\mathfrak{n}$ is $Ad\,K$-irreducible. Similarly, b_a^j was used before to denote the eigenvalue of $C_{\mathfrak{p}_a}$, if this operator was scalar, on $\mathfrak{n}_j$, while in here b_a^ϕ is the eigenvalue of this same operator on $\mathfrak{n}^j$. No confusion should arise from this since we shall use this second notation only when $\mathfrak{n}$ is an irreducible $Ad\,K$-module.*

The necessary condition for existence of an adapted Einstein metric on M given in Corollary 1.4 can now be rewritten as follows:

Corollary 3.1. *If there exists on M an Einstein adapted metric, then there are positive constants $\lambda_1, \dots, \lambda_s$ such that*

$$\sum_{a=1}^{s} \lambda_a (b_a^{\phi_1} - b_a^{\phi_2}) = 0,$$

for every $\phi_1, \phi_2 \in \mathcal{R}_\mathfrak{n}$.

The condition in Corollary 3.1 shall play a fundamental role as a preliminary test for existence of Einstein adapted metrics. It is a very restrictive condition which is not satisfied by many of the spaces under study in this Chapter.

For any roots ϕ and α let $\phi + n\alpha$, $p_{\alpha\phi} \leq n \leq q_{\alpha\phi}$, be the α-series containing ϕ. By definition, the α-series containing ϕ is the set of all roots of the form $\phi + n\alpha$ where n is an integer. It is known that $\phi + n\alpha$ is an interrupted series ([16], Chap.III, §4). For roots α and ϕ the square of the structure constant $N_{\alpha\phi}$ is given by

$$N_{\alpha,\phi}^2 = \frac{q_{\alpha\phi}(1 - p_{\alpha\phi})}{2} \alpha(H_\alpha). \tag{3.17}$$

Proposition 3.1. *Suppose that $rank\,L = rank\,G$. For every $\phi \in \mathcal{R}_\mathfrak{n}$ and $a = 1, \dots, s$,*

$$b_a^\phi = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_{\mathfrak{p}_a}^+} d_{\alpha\phi} |\alpha|^2,$$

where $d_{\alpha\phi} = q_{\alpha\phi} - p_{\alpha\phi} - 2p_{\alpha\phi}q_{\alpha\phi}$ and $\phi + n\alpha$, $p_{\alpha\phi} \leq n \leq q_{\alpha\phi}$ is the α-series containing ϕ.

Proof: By using (3.16) and (3.11) we obtain the following:

$$\begin{aligned}
b_a^\phi = &\ Kill(C_{\mathfrak{p}_a^{\mathbb{C}}} E_\phi, E_{-\phi}) \\
= &\ \textstyle\sum_{\alpha\in\mathcal{R}_{\mathfrak{p}_a}} Kill([E_{-\alpha},[E_\alpha,E_\phi]],E_{-\phi}) \\
= &\ \textstyle\sum_{\alpha\in\mathcal{R}_{\mathfrak{p}_a}} N_{\alpha,\phi} Kill([E_{-\alpha},E_{\phi+\alpha}],E_{-\phi}) \\
= &\ \textstyle\sum_{\alpha\in\mathcal{R}_{\mathfrak{p}_a}} N_{\alpha,\phi} N_{-\alpha,\phi+\alpha} Kill(E_\phi,E_{-\phi}) \\
= &\ \textstyle\sum_{\alpha\in\mathcal{R}_{\mathfrak{p}_a}} N_{\alpha,\phi} N_{-\alpha,\phi+\alpha}
\end{aligned}$$

From (3.4) we have $N_{-\alpha,\phi+\alpha} = N_{\alpha,\phi}$ and we get

$$b_a^\phi = \sum_{\alpha\in\mathcal{R}_{\mathfrak{p}_a}} N_{\alpha,\phi}^2 = \sum_{\alpha\in\mathcal{R}_{\mathfrak{p}_a}^+} (N_{\alpha,\phi}^2 + N_{-\alpha,\phi}^2).$$

Now let $\phi + n\alpha$, $p_{\alpha\phi} \le n \le q_{\alpha\phi}$, be the α-series containing ϕ. It is known that

$$N_{\alpha,\phi}^2 = \frac{q_{\alpha\phi}(1-p_{\alpha\phi})}{2}\alpha(H_\alpha),$$

as mentioned in (3.17).
On the other hand, to compute $N_{-\alpha,\phi}^2$ we need the $(-\alpha)$-series containing ϕ. Clearly, this series is $\phi - n'\alpha$, where $-q_{\alpha\phi} \le n' \le -p_{\alpha\phi}$. Hence, we obtain the following:

$$N_{-\alpha,\phi}^2 = \frac{-p_{\alpha\phi}(1-(-q_{\alpha\phi}))}{2}(-\alpha)(H_{-\alpha}) = \frac{-p_{\alpha\phi}(1+q_{\alpha\phi})}{2}\alpha(H_\alpha).$$

Hence,

$$b_a^\phi = \sum_{\alpha\in\mathcal{R}_{\mathfrak{p}_a}^+} \left(\frac{q_{\alpha\phi}(1-p_{\alpha\phi})}{2} - \frac{-p_{\alpha\phi}(1+q_{\alpha\phi})}{2}\right)\alpha(H_\alpha),$$

which yields the required formula.
□

Let us consider a decomposition of $\mathfrak{k}$ into its center $\mathfrak{k}_0$ and simple ideals $\mathfrak{k}_a$, for $a = 1,\dots,t$,

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_t, \tag{3.18}$$

and let γ_a denote the eigenvalue of the Casimir operator of $\mathfrak{k}$ on $\mathfrak{k}_a$. We present a formula to compute the eigenvalues γ_a's by making use of dual Coxeter numbers. We start by recalling some facts about roots. On this topic we refer to ([13], V.5) and ([17], 10.4).

There are at most two different lengths in a given irreducible root system, and the corresponding roots are designated by **long** and **short** roots. If there is only one length it is conventional to say that all the roots are long. If α is a long root and β is short, then

$$\begin{aligned} \frac{|\alpha|^2}{|\beta|^2} &= 3, \quad \text{in the case of } G_2 \text{ and} \\ \frac{|\alpha|^2}{|\beta|^2} &= 2, \quad \text{in the case of } B_n, C_n \text{ and } F_4. \end{aligned} \tag{3.19}$$

In the remaining cases, A_n, D_n, E_6, E_7 and E_8, there is only one length. These facts can be read off from the corresponding Dynkin diagrams.
We also recall that a length of a root α is given by

$$|\alpha|^2 = \alpha(H_\alpha) = Kill(H_\alpha, H_\alpha).$$

The **dual Coxeter number** of a simple Lie algebra $\mathfrak{g}$ is the number given by

$$h^*(\mathfrak{g}) = \frac{1}{|\alpha|^2},$$

where α is a long root (see e.g. [35]). The dual Coxeter numbers of each irreducible root system are given in Table 3.1.
We may suppose that $\mathfrak{h}_a = \mathfrak{h} \cap \mathfrak{k}_a$ is a Cartan subalgebra of $\mathfrak{k}_a$ and thus a root of $\mathfrak{k}_a$ can be viewed as a root for $\mathfrak{g}$. Hence we can compare lengths of roots of $\mathfrak{g}$ with lengths of roots of $\mathfrak{k}_a$. So let δ_a be the ratio of the square length of a long root for $\mathfrak{g}$ to that of $\mathfrak{k}_a$, i.e.,

$$\delta_a = \frac{|\alpha|^2_{\mathfrak{g}}}{|\beta|^2_{\mathfrak{g}}} = \frac{Kill(H_\alpha, H_\alpha)}{Kill(H_\beta, H_\beta)},$$

where α is a long root of $\mathfrak{g}$ and β is a long root of $\mathfrak{k}_a$. Clearly, $\delta_a = 1$ if there exists only one length for $\mathfrak{g}$ or if both $\mathfrak{g}$ and $\mathfrak{k}_a$ have two lengths. If $\delta_a \neq 1$, then, according to (3.19), δ_a is equal to either 2 or 3. We recall the following result by D. Panyushev:

Proposition 3.2. *[35] Suppose that $\mathfrak{g}$ is simple. Then*

$$\gamma_a = \frac{h^*(\mathfrak{k}_a)}{\delta_a . h^*(\mathfrak{g})}, \; a = 1, \ldots, s,$$

where $h^*(\mathfrak{k}_a)$ and $h^*(\mathfrak{g})$ are the dual Coxeter numbers of $\mathfrak{k}_a$ and $h^*(\mathfrak{g})$, respectively.

We observe that so far we used only the fact that L has maximal rank. Now if, in addition, $F = K/L$ is a symmetric space, then we consider its DeRham decomposition

$$K/L = K_1/L_1 \times \ldots \times K_s/L_s, \tag{3.20}$$

where, for $a = 1, \ldots, s$, K_a is simple, as in Section 2.5. We observe that since L has maximal rank in the deRham decomposition of K/L the factor K_0/L_0 must be trivial. The Lie algebras of $K_1, \ldots, K_s$, denoted by $\mathfrak{k}_1, \ldots, \mathfrak{k}_s$, are some of the ideals in the decomposition (3.18). As explained at the beginning of Section 2.5, the irreducible $Ad\,L$-submodules $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are chosen as the symmetric reductive complements of $\mathfrak{l}_a$ in $\mathfrak{k}_a$, for $a = 1, \ldots, s$. Hence, the constant γ_a defined throughout by the equality $Kill_{\mathfrak{k}} \mid_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a Kill \mid_{\mathfrak{p}_a \times \mathfrak{p}_a}$ in previous sections, is now just the eigenvalue of the Casimir operator of $\mathfrak{k}$ on $\mathfrak{k}_a$, and thus can be determined by the formula in Proposition 3.2.
In Appendix A we compute the eigenvalues b_a^ϕ's and γ_a's using Propositions 3.1 and 3.2. Their values are indicated in Tables 3.4, 3.5, 3.6 and 3.7 in Section 3.6 for each bisymmetric triple.

3.2 Bisymmetric Triples of Maximal Rank - Classification

Let us consider a homogeneous fibration $F \to M \to N$, for $M = G/L$, $N = G/K$ and $F = K/L$, where G is a compact connected semisimple Lie group and $L \subsetneq K \subsetneq G$ connected closed non-trivial subgroups such that F and N are symmetric spaces. We shall call such a fibration a **bisymmetric fibration**. In particular, the pairs (G, K) and (K, L) are symmetric pairs of compact type [16]. With a slight abuse of terminology, we shall also say that the pairs of Lie algebras $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{k}, \mathfrak{l})$ are symmetric pairs of compact type whenever the corresponding pairs (G, K) and (K, L) are.

Definition 3.1. *A* ***bisymmetric triple*** *is a triple* $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ *where* $\mathfrak{g}$*,* $\mathfrak{k}$ *and* $\mathfrak{l}$ *are Lie algebras satisfying the following conditions:*
(i) $\mathfrak{l} \subsetneq \mathfrak{k} \subsetneq \mathfrak{g}$*;*
(ii) $(\mathfrak{g}, \mathfrak{k})$ *and* $(\mathfrak{k}, \mathfrak{l})$ *are symmetric pairs of compact type.*
A bisymmetric triple is said to be irreducible if $\mathfrak{g}$ *is a simple Lie algebra and said to be of maximal rank if* $\mathfrak{l}$ *has maximal rank in* $\mathfrak{g}$*, i.e., it contains a maximal toral subalgebra of* $\mathfrak{g}$*.*

Clearly, there is a one-to-one correspondence between bisymmetric fibrations, up to cover, and bisymmetric triples. **All the bisymmetric triples** $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ **considered in this chapter are irreducible and of maximal rank**, even when this is not explicitly stated. Consequently, **any bisymmetric fibration**

$\mathbf{F} \to \mathbf{M} \to \mathbf{N}$ here considered is such that $\mathbf{L}$ is a subgroup of maximal rank in $\mathbf{G}$ and $\mathbf{N} = \mathbf{G}/\mathbf{K}$ is an irreducible symmetric space.

Definition 3.2. *A bisymmetric triple* $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ *is said to be of*
(i) ***Type I*** *if F is an isotropy irreducible symmetric space; equivalently, if $\mathfrak{p}$ is an irreducible $\mathrm{Ad}\, L$-module;*
(ii) ***Type II*** *if F is the direct product of two isotropy irreducible symmetric spaces; equivalently, if $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are nontrivial irreducible $\mathrm{Ad}\, L$-modules.*

A bisymmetric fibration $F \to M \to N$ of is said to be of Type I or II if the corresponding bisymmetric triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ is either of Type I or II, respectively.

As we shall see any irreducible bisymmetric triple of maximal rank with $\mathfrak{g}$ simple is of Type I or II.

Isotropy irreducible symmetric spaces have been classified and a classification can be found in [16]. By using this we obtain a list of all possible triples $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ such that $\mathfrak{l}$ and $\mathfrak{k}$ are subalgebras of maximal rank of $\mathfrak{g}$ and $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{k}, \mathfrak{l})$ are symmetric pairs of compact type. By inspection of the classification of symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ of compact type in [16] we obtain that those of maximal rank are the pairs in Tables 3.2 and 3.3.
We observe that the cases when $\mathfrak{k}$ is the centralizer of a torus are only the cases $(\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathbb{R})$, $(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R})$, $(\mathfrak{so}_{2n}, \mathfrak{u}_n)$, $(\mathfrak{so}_n, \mathbb{R} \oplus \mathfrak{so}_{n-2})$, $(\mathfrak{sp}_n, \mathfrak{u}_n)$ and $(\mathfrak{su}_n, \mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R})$. This follows from the fact that these are the only subalgebras $\mathfrak{k}$ corresponding to painted Dynkin diagrams of the Dynkin diagram of $\mathfrak{g}$ or as they are the only ones such that $\mathfrak{k}$ is not centerless. In all the other cases $\mathfrak{k}$ is semisimple. If $\mathfrak{k}$ is simple then $(\mathfrak{k}, \mathfrak{l})$ shall be an irreducible symmetric pair, i.e., $\mathfrak{p}$ is an irreducible L-invariant subspace. Thus, $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ is of type I. In the cases where $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathbb{R}$ with $\mathfrak{k}_1$ a simple ideal of $\mathfrak{k}$, since we require $\mathfrak{l}$ to be of maximal rank, we have $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathbb{R}$, where $\mathfrak{l}_1$ is a subalgebra of $\mathfrak{k}_1$ with maximal rank and $(\mathfrak{k}, \mathfrak{l}) \cong (\mathfrak{k}_1, \mathfrak{l}_1)$ is an irreducible symmetric pair. Thus, in this case, $\mathfrak{p} \cong \mathfrak{p}_1$ is also an irreducible L-invariant subspace and $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ is of type I. In the cases where $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$, with $\mathfrak{k}_1$ and $\mathfrak{k}_2$ simple ideals of $\mathfrak{k}$, we have $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$, where, for $i = 1, 2$, $\mathfrak{l}_i$ is a subalgebra of $\mathfrak{k}_i$ of maximal rank. Clearly, one of the $\mathfrak{l}_i$'s must be proper as we require that $\mathfrak{l}$ is a proper subalgebra of $\mathfrak{k}$. If both $\mathfrak{l}_1$ ad $\mathfrak{l}_2$ are proper, then $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are nonzero irreducible $\mathfrak{l}$-invariant subspaces. Hence, in this case, $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ is of type II. If exactly one of $\mathfrak{l}_i$'s coincides with $\mathfrak{k}_i$, then $(\mathfrak{k}, \mathfrak{l}) \cong (\mathfrak{k}_j, \mathfrak{l}_j)$ and $\mathfrak{p} = \mathfrak{p}_j$, for that j satisfying $\mathfrak{l}_j \neq \mathfrak{k}_j$, and once again $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ is of type I. Finally, we have the case of the spaces $(\mathfrak{su}_n, \mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R})$,

$p = 1, \ldots, n-1$. Clearly, $\mathfrak{l}$ must be of the form $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \mathbb{R}$, where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are maximal rank subalgebras of $\mathfrak{su}_p$ and $\mathfrak{su}_{n-p}$, respectively. We obtain a triple of type I if exactly one of the $\mathfrak{l}_i$'s is proper and a triple of type II if both $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are proper. This proves the following:

Lemma 3.1. *An irreducible bisymmetric triple of maximal rank* $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ *such that* $\mathfrak{g}$ *is simple is either of Type I or II. Moreover, all such bisymmetric triples* $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ *are those in Tables and 3.4, 3.5, 3.6 and 3.7.*

Lemma 3.2. *For an irreducible bisymmetric triple of maximal rank* $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ *of Type I, let* γ *be the eigenvalue of the Casimir operator of* $\mathfrak{k}$ *on* $\mathfrak{p}$ *and* b^{ϕ}*'s the eigenvalues of the Casimir operator of* $\mathfrak{p}$ *on* $\mathfrak{n}$*. For each bisymmetric triple of maximal rank these eigenvalues have the values listed in Tables 3.4 and 3.5.*

Lemma 3.3. *For an irreducible bisymmetric triple of maximal rank* $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ *of Type II, let* γ_a *be the eigenvalue of the Casimir operator of* $\mathfrak{k}$ *on* $\mathfrak{p}_a$ *and* b_a^{ϕ}*'s the eigenvalues of the Casimir operator of* $\mathfrak{p}_a$ *on* $\mathfrak{n}$*,* $a = 1, 2$*. For each bisymmetric triple of maximal rank these eigenvalues have the values listed in Tables 3.6 and 3.7.*

3.3 Einstein Adapted Metrics for Type I

In this Section we determine all the bisymmetric Riemannian fibrations $F \to M \to N$ of maximal rank of Type I which admit an Einstein adapted metric. We recall that for Type I, $\mathfrak{p}$ is an irreducible $Ad\, L$-submodule. Moreover, since $\mathfrak{n}$ is an irreducible $Ad\, K$-module, any adapted metric is binormal. As in addition F and N are symmetric spaces, we may apply Corollary 2.9. We recall that according to Corollary 2.9, there exists on M an Einstein (binormal) adapted metric if and only if

$$\begin{aligned} &(i) \text{ the Casimir operator of } \mathfrak{p} \text{ is scalar on } \mathfrak{n} \text{ and} \\ &(ii) \triangle = 1 - 2\gamma(1 - \gamma + 2b) \geq 0. \end{aligned} \tag{3.21}$$

If these two conditions are satisfied, Einstein binormal metrics are, up to homothety, given by

$$g_M = B\,|_{\mathfrak{p}\times\mathfrak{p}} \oplus XB\,|_{\mathfrak{n}\times\mathfrak{n}}, \text{ where } X = \frac{1 \pm \sqrt{\triangle}}{2\gamma}. \tag{3.22}$$

We also recall that b is the eigenvalue of $C_{\mathfrak{p}}$ on $\mathfrak{n}$, in the case when this operator is scalar, and γ is the eigenvalue of the Casimir operator of $\mathfrak{k}$ on $\mathfrak{p}$. These constants are computed in Appendix A and their values are indicated in Tables 3.4 and

3.5, as has been stated in Lemma 3.2. We recall that condition (i) translates into $b^\phi = b$, for every $\phi \in \mathcal{R}_\mathfrak{n}$, for Type I, according to Corollary 3.1. Hence, the first test for existence of an adapted Einstein metric shall be to observe if there exists only one eigenvalue b^ϕ in the corresponding columns of Tables 3.4 and 3.5.

Theorem 3.1. *The bisymmetric fibrations $F \to M \to N$ of Type I such that there exists on M an Einstein adapted metric are those whose bisymmetric triples are listed in Tables 3.8 and 3.9. For each case there are exactly two Einstein adapted metrics. Furthermore, these Einstein metrics are, up to homothety, given by*

$$g_M = B\mid_{\mathfrak{p}\times\mathfrak{p}} \oplus XB\mid_{\mathfrak{n}\times\mathfrak{n}},$$

where X is indicated in the Tables mentioned above.

Proof: As explained in the discussion above, by Corollary 2.9, the existence of an adapted Einstein metric implies that the Casimir operator of $\mathfrak{p}$ is scalar on $\mathfrak{n}$. By inspection we conclude from Tables 3.5 and 3.4 that the only spaces satisfying this condition are those corresponding to the labels

$A.6$, $A.13$, $A.21$, $A.25$, $A.26$ $A.31$, $A.32$, $A.34$, $A.36$, $A.41$, $A.43$, $A.46$, $A.52$, $A.53$

and
A.1 for $l = \frac{p}{2}$ with p even; in this case, $b = \frac{p}{4n}$;
A.5 for $s = \frac{n-p}{2}$ with $n-p$ even; in this case, $b = \frac{n-p}{2(2n-1)}$;
A.10 for $l = \frac{p}{2}$ with p even; in this case, $b = \frac{p}{4(n-1)}$;
A.18 for $l = \frac{p}{2}$ with p even; in this case, $b = \frac{p}{8(n+1)}$;
A.50 for $p = 1$; in this case $b = \frac{1}{9}$;
A.54 for $p = 1$; in this case $b = \frac{1}{8}$.

We compute $\triangle$ given in formula 3.21, and the values obtained are as follows

	$\triangle$
$A.1$	$\left(\frac{n-p}{n}\right)^2 > 0$
$A.5$	$\frac{4p^2+8p-4n+5}{(2n-1)^2} > 0, \forall p = \lfloor\frac{\sqrt{4n-1}}{2}\rfloor, \dots, n-1$
$A.6$	$\left(\frac{2p+1}{2n-1}\right)^2 > 0$
$A.10$	$\frac{p^2-(2n+1)p+n^2+1}{(n-1)^2} > 0$
$A.13$	$\left(\frac{p-n}{n-1}\right)^2 > 0$
$A.18$	$\frac{3p^2+(3-4n)p+2(n^2+1)}{2(n+1)^2} > 0$
$A.21$	$\left(\frac{n-p}{n+1}\right)^2 > 0$
$A.25$	$\frac{106-63p+7p^2}{162} > 0$, iff $p = 1$, 7
$A.26$	$\frac{49}{81} > 0$
$A.31$	$\frac{1}{4} > 0$
$A.32$	$\frac{11}{18} > 0$
$A.34$	$\frac{7p^2-56p+113}{225} > 0$
$A.36$	$\frac{196}{225} > 0$
$A.41$	$-\frac{2}{25} < 0$
$A.43$	$\frac{64}{81} > 0$
$A.46$	$\frac{164-60p+5p^2}{324} > 0$, $p = 2, 4$
$A.50$	$\frac{25}{81} > 0$
$A.52$	$\frac{1}{9} > 0$, $p = 2$; $-\frac{1}{9}$, $p = 4$
$A.53$	$\frac{25}{36} > 0$
$A.54$	$\frac{1}{4} > 0$

For $A.25$, since $p = 1, 3, 5, 7$, then $\triangle > 0$ for $p = 1, 7$ and $\triangle < 0$ otherwise.
For $A.46$, we have $p = 2, 4, 6$. Then, $\triangle > 0$ for $p = 2, 4$ and $\triangle < 0$ for $p = 6$.
For $A.34$, we have $p = 1, \dots, 4$ and thus $\triangle > 0$ for every p.
For $A.10$, $p = 1, \dots, \lfloor\frac{n}{2}\rfloor$. We have that $\triangle \geq 0$ if and only if

$$p \in \Bigg(\big(-\infty, \frac{2n+1-\sqrt{4n-3}}{2}\big) \cup \big(\frac{2n+1+\sqrt{4n-3}}{2}, +\infty\big)\Bigg) \cap \{1, \dots, \lfloor\frac{n}{2}\rfloor\}$$

We can show that $\frac{2n+1-\sqrt{4n-3}}{2} \geq \frac{n}{2}$ and thus $\triangle > 0$, for every $p = 1, \dots, \lfloor\frac{n}{2}\rfloor$.
For $A.5$, $p = 1, \dots, n-1$. We show that $\triangle \geq 0$, if and only if

$$p \in \left(-1 + \frac{\sqrt{4n-1}}{2}, +\infty\right) \cap \{1, \dots, n-1\}$$

Since, for every n, $-1 + \frac{\sqrt{4n-1}}{2} < n-1$ we conclude that $\triangle > 0$, if and only if $p = \lfloor -1 + \frac{\sqrt{4n-1}}{2}\rfloor + 1, \dots, n-1 = \lfloor\frac{\sqrt{4n-1}}{2}\rfloor, \dots, n-1$.

Finally, we compute $X = \frac{1\pm\sqrt{\triangle}}{2\gamma}$ for those cases when $\triangle > 0$. The values of X are indicated in Table 3.8.

$\square$

Remark 3.2. *For the bisymmetric triples of Type I such that $C_{\mathfrak{p}}$ is scalar on $\mathfrak{n}$ but $\triangle < 0$, there is still a complex non real solution $X = \frac{1\pm\sqrt{\triangle}}{2\gamma}$. So even in these cases we can conclude that there exists a complex Einstein adapted metric on M as in (3.22). These are just the cases A.5 for $p \geq \lfloor -1 + \frac{\sqrt{4n-1}}{2} \rfloor$, A.25 for $p = 3, 5$ and A.41.*

3.4 Einstein Adapted Metrics for Type II

In this Section we study the existence of Einstein adapted metrics on bisymmetric fibrations of Type II. Whereas for Type I any adapted metric was binormal this is clearly not true for Type II, since $\mathfrak{p}$ is not an irreducible $Ad\,L$-module. We shall classify all the bisymmetric triples which admit an Einstein binormal metric. Since for bisymmetric fibrations, in particular, F is a symmetric space, we know from Corollary 2.11 that for any Einstein binormal metric g_M, g_F is also Einstein. We shall also classify all the bisymmetric triples which admit an Einstein non-binormal adapted metric g_M whose restriction g_F is also Einstein.

Since for type II $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2$, where $\mathfrak{p}_i$, $i = 1, 2$, are irreducible L-modules and $\mathfrak{n}$ is an irreducible K-module, the existence of an Einstein adapted metric implies that there exist positive constants λ_1, λ_2, such that the operator $\lambda_1 C_{\mathfrak{p}_1} + \lambda_2 C_{\mathfrak{p}_2}$ is scalar on $\mathfrak{n}$, according to Corollary 1.4. As rephrased in Corollary 3.1, the condition above translates into

$$\lambda_1(b_1^{\phi_1} - b_1^{\phi_2}) + \lambda_2(b_2^{\phi_1} - b_2^{\phi_2}) = 0, \text{ for every } \phi_1, \phi_2 \in \mathcal{R}_{\mathfrak{n}} \tag{3.23}$$

By using Tables 3.6 and 3.7 we conclude the following:

Lemma 3.4. *The only bisymmetric triples satisfying condition (3.23) are the cases A.15, A.23, A.33, A.42, A.47 and*

$A.3$ *for p=2l, n-p=2s*

$A.12$ *for p=2l, n-p=2s*

$A.16$ *for p=2l*

$A.20$ *for p=2l, n-p=2s*

$A.24$ *for p=2l*

$A.55$ *for p=1*

For all other bisymmetric triples of Type II we can conclude that there exists no Einstein adapted metric on M.

Furthermore, for all the triples listed in Lemma 3.4, we observe that $C_{\mathfrak{p}_1}$ and $C_{\mathfrak{p}_2}$ are scalar on $\mathfrak{n}$ and thus $C_{\mathfrak{p}}$ is also scalar on $\mathfrak{n}$. We shall write

$$C_{\mathfrak{p}_i}\mid_{\mathfrak{n}}= b_i Id_{\mathfrak{n}},\; i=1,2 \tag{3.24}$$

$$C_{\mathfrak{p}}\mid_{\mathfrak{n}}= b Id_{\mathfrak{n}},\ \text{for } b=b_1+b_2, \tag{3.25}$$

following the notation used in previous chapters.

Theorem 3.2. *The bisymmetric fibrations $F\to M\to N$ of Type II such that there exists on M an Einstein binormal metric are those whose bisymmetric triples are listed in Table 3.10. Furthermore, the binormal Einstein metrics are, up to homothety, given by*

$$g_M = B\mid_{\mathfrak{p}\times\mathfrak{p}} \oplus XB\mid_{\mathfrak{n}\times\mathfrak{n}},$$

where X is indicated in Table 3.10. In all the cases, g_N and g_F are also Einstein.

Proof: Binormal Einstein metrics are in this case given by Corollary 2.10. First we observe that in order to exist an Einstein binormal metric on M, $C_{\mathfrak{p}}$ must be scalar on $\mathfrak{n}$ and $C_{\mathfrak{k}}$ must be scalar on $\mathfrak{p}$. The triples which satisfy the first condition are those listed in Lemma 3.4. Furthermore, the second condition implies that $\gamma_2=\gamma_1$. From the cases in Lemma 3.4 we conclude from Tables 3.6 and 3.7 that the spaces which satisfy the condition $\gamma_2=\gamma_1$ are those listed below:

A.3 for $s=l=2p$, $n=4l$; in this case, $\gamma_1=\gamma_2=\frac{1}{2}$ and $b_1=b_2=\frac{1}{8}$;

A.12 for $s=l=2p$, $n=4l$, $l\geq 2$; in this case, $\gamma_1=\gamma_2=\frac{2l-1}{4l-1}$ and $b_1=b_2=\frac{l}{2(4l-1)}$;

A.15 for $n=2p$, $p\geq 2$; in this case, $\gamma_1=\gamma_2=\frac{p-1}{2p-1}$ and $b_1=b_2=\frac{p-1}{4(2p-1)}$;

A.16 for $p=2l$, $n=4l$; in this case, $\gamma_1=\gamma_2=\frac{2l-1}{4l-1}$ and $b_1=\frac{l}{2(4l-1)}$, $b_2=\frac{2l-1}{4(4l-1)}$;

A.20 for $s=l=2p$, $n=4l$; in this case, $\gamma_1=\gamma_2=\frac{2l+1}{4l+1}$ and $b_1=b_2=\frac{l}{4(4l+1)}$;

A.23 for $n=2p$; in this case, $\gamma_1=\gamma_2=\frac{p+1}{2p+1}$ and $b_1=b_2=\frac{p+1}{4(2p+1)}$;

A.24 for $p=2l$, $n=4l$; in this case, $\gamma_1=\gamma_2=\frac{2l+1}{4l+1}$ and $b_1=\frac{l}{4(4l+1)}$, $b_2=\frac{2l+1}{4(4l+1)}$.

Now Einstein binormal metrics are given by positive solutions of (2.32). Since $c_{\mathfrak{k},\mathfrak{n}}=\frac{1}{2}$, we have that there exists an Einstein binormal metric if and only if

$$\triangle = 1-2\gamma(1-\gamma+2b)\geq 0, \tag{3.26}$$

where $\gamma = \gamma_1 = \gamma_2$ and b is the eigenvalue of $C_{\mathfrak{p}}$ on $\mathfrak{n}$, i.e., $b = b_1 + b_2$. In such a case these Einstein metrics are given by homotheties of

$$g_M = B\mid_{\mathfrak{p}\times\mathfrak{p}} \oplus XB\mid_{\mathfrak{n}\times\mathfrak{n}}, \text{ where } X = \frac{1 \pm \sqrt{\triangle}}{2\gamma}.$$

We compute $\triangle$:

	$\triangle$
$A.3$	0
$A.12$	$\frac{1}{(4l-1)^2} > 0$
$A.15$	$\frac{1}{2p-1} > 0$
$A.16$	$\frac{2l}{(4l-1)^2} > 0$
$A.20$	$\frac{4l^2+2l+1}{(4l+1)^2} > 0$
$A.23$	$-\frac{1}{2p+1} < 0$
$A.24$	$\frac{l(2l-1)}{(4l+1)^2} > 0$

Except in the case A.23, there exists an Einstein adapted metric. The values for X are indicated in Table 3.10.
g_N is Einstein because N is irreducible and g_F is Einstein due to Corollary 2.11.
□

Remark 3.3. *In the case A.23, where $C_{\mathfrak{p}}$ is scalar on $\mathfrak{n}$ but $\triangle = -\frac{1}{2p+1} < 0$, for every p, we can still consider the non-real complex solution $X = \frac{1\pm\sqrt{\triangle}}{2\gamma}$ which gives rise to a non-real complex Einstein binormal metric on M.*

Theorem 3.3. *The bisymmetric fibrations $F \to M \to N$ of Type II such that there exists on M an Einstein adapted metric such that g_F is Einstein are those with an Einstein binormal metric, as in Theorem 3.2 and Table 3.10, and the fibration corresponding to the bisymmetric triple*

$$(\mathfrak{su}_{2(l+s)}, \mathfrak{su}_{2l} \oplus \mathfrak{su}_{2s} \oplus \mathbb{R}, \mathfrak{su}_l \oplus \mathfrak{su}_l \oplus \mathfrak{su}_s \oplus \mathfrak{su}_s \oplus \mathbb{R}^3),$$

whose Einstein adapted metric is, up to homothety, given by

$$g_M = \frac{2l}{l+s} B\mid_{\mathfrak{p}_1\times\mathfrak{p}_1} \oplus \frac{2s}{l+s} B\mid_{\mathfrak{p}_2\times\mathfrak{p}_2} \oplus B\mid_{\mathfrak{n}\times\mathfrak{n}}.$$

This metric is binormal if and only if $l = s$.

Proof: The only cases which may admit an Einstein adapted metric are those listed in Lemma 3.4, since they are the only that satisfy the necessary condition (3.23) for existence of such a metric. Furthermore, if we require that g_F is also

Einstein, then we must have one of the two conditions in Corollary ??. In the first case, $\gamma = 1 = \gamma_2$, g_M is a binormal metric according to Corollary ??. These are the cases given by Theorem 3.2 and listed in Table 3.10. In the second case, we have $\gamma_2 = 1 - \gamma_1$. By inspection of the triples listed in Lemma 3.4, we conclude that this is possible only for the triple A.3 when $p = 2l$ and $n - p = 2s$. In this case $\gamma_1 = \frac{l}{l+s}$, $\gamma_2 = \frac{s}{l+s} = 1 - \gamma_1$, $b_1 = \frac{\gamma_1}{4}$ and $b_2 = \frac{\gamma_2}{4} = \frac{1-\gamma_1}{4}$. Using Corollary 2.13, we obtain $D(\gamma_1) = 0$ and thus $X_1 = \frac{l+s}{2l}$, $X_2 = \frac{l+s}{2s}$.
□

For bisymmetric fibrations of Type II the classifications of all Einstein adapted metrics is a difficult problem due to the high complexity of the Einstein equations. The classification of Einstein binormal metrics and Einstein adapted metrics such that g_F is Einstein as well, as done above, is a way to restrict the problem in a way that the Einstein equations are manageable. It can be seen by Theorems 3.3 and 3.2 that no bisymmetric fibration of Type II in the exceptional admits neither an Einstein binormal metric nor an Einstein adapted metric with g_F Einstein. However, for this spaces of exceptional type is is possible to classify all Einstein adapted metrics with the help of Maple. Once again, since $C_{\mathfrak{p}_1}$ and $C_{\mathfrak{p}_2}$ must be scalar on $\mathfrak{n}$, from Lemma 3.4 we know that the only cases of Type II and exceptional are the fibrations A.55 for $p = 1$, A.33, A.47 and A.42. The results obtained for these cases are synthetized in Theorem 3.4 and Table 3.6.

Theorem 3.4. *The only bisymmetric fibrations $F \to M \to N$ of Type II, such that G is an exceptional Lie group, which admit an Einstein adapted metric are those whose bisymmetric triples are listed in Table 3.6. None of these metrics is neither binormal nor such that g_F is Einstein.*

Proof: As mentioned above, it follows from Lemma 3.4 that the only cases of Type II with $\mathfrak{g}$ exceptional wich may admit an Einstein adapted metric are the cases A.55 for $p = 1$, A.33, A.47 and A.42. We recall that for each of these spaces any Einstein adapted metric is of the form

$$g_M = \frac{1}{X_1} B \mid_{\mathfrak{p}_1 \times \mathfrak{p}_1} \oplus \frac{1}{X_2} B \mid_{\mathfrak{p}_2 \times \mathfrak{p}_2} \oplus B \mid_{\mathfrak{n} \times \mathfrak{n}},$$

where X_1 and X_2 are positive solutions of the system of equations given in Theorem 2.5 which are as follows:

$$2\gamma_1 X_1^2 X_2 + (1 - \gamma_1) X_2 - 2\gamma_2 X_1 X_2^2 - (1 - \gamma_2) X_1 = 0, \tag{3.27}$$

$$2b_1 X_2 + 2b_2 X_1 - 2X_1 X_2 + 2\gamma_1 X_1^2 X_2 + (1 - \gamma_1) X_2 = 0. \tag{3.28}$$

Also we recall that the eigenvalues b_i and γ_i, for $i = 1, 2$, can be found in Table 3.6. To show the result we use Maple and so many details are ommited.

A.33) For the bisymmetric triple A.33 the non-zero solutions of the equations (3.27) and (3.28) are of the form

$$X_1 = \alpha_i,\ X_2 = -\frac{7}{4}\alpha_i^3 + 12\alpha_i^2 - \frac{5899}{36}\alpha_i + 19,$$

where α_i is a root of the polynomial

$$t(z) = 63z^4 - 432z^3 + 1088z^2 - 1224z + 513.$$

Since $X_1 = \alpha_i$ we are interested only in positive real roots of t. This polynomial has exactly two positive roots which are

$$\alpha_1 = \tfrac{12\xi\beta}{7} + \tfrac{\beta^3}{126} - \tfrac{\sqrt{6}}{126}\tfrac{(320\xi^2\beta^2+7\xi^4\beta^2-25781\beta^2+43416\xi^3)^{\frac{1}{2}}}{\xi\beta}$$
$$\alpha_2 = \tfrac{12\xi\beta}{7} + \tfrac{\beta^3}{126} + \tfrac{\sqrt{6}}{126}\tfrac{(320\xi^2\beta^2+7\xi^4\beta^2-25781\beta^2+43416\xi^3)^{\frac{1}{2}}}{\xi\beta}$$

where $\xi = (17756 + 81\sqrt{7662443})^{\frac{1}{6}}$ and $\beta = (960\xi^2 - 42\xi^4 + 154686)^{\frac{1}{4}}$.
Simple calculations show that α_i, for $i = 1, 2$, yields positive real values for X_2 as well. Approximations for the corresponding values of X_1 and X_2 are given below:

i	X_1	X_2
1	0.5526	3.6958
2	0.7432	4.7185

Hence, there are on M exactly two Einstein adapted metrics.

A.55, for $p = 1$**)** In this case the non-zero solutions of the equations (3.27) and (3.28) are given by

$$X_1 = \alpha_i,\ X_2 = -\frac{156}{7}\alpha_i^3 + \frac{552}{7}\alpha_i^2 - \frac{571}{7}\alpha_i + \frac{176}{7},$$

where α_i is a root of the polynomial

$$t(z) = 234z^4 - 828z^3 + 993z^2 - 474z + 77.$$

The polynomial t has exactly four positive roots which are indicated below:

$$\alpha_1 = \tfrac{23}{26} - \tfrac{\sqrt{2}\beta}{156} - \tfrac{1}{156}\left(\tfrac{-3664\xi\beta+26\xi^2\beta+71786\beta+26460\sqrt{2}\xi}{\xi\beta}\right)^{\frac{1}{2}}$$
$$\alpha_2 = \tfrac{23}{26} - \tfrac{\sqrt{2}\beta}{156} + \tfrac{1}{156}\left(\tfrac{-3664\xi\beta+26\xi^2\beta+71786\beta+26460\sqrt{2}\xi}{\xi\beta}\right)^{\frac{1}{2}}$$
$$\alpha_3 = \tfrac{23}{26} + \tfrac{\sqrt{2}\beta}{156} - \tfrac{1}{156}\left(\tfrac{-3664\xi\beta+26\xi^2\beta+71786\beta+26460\sqrt{2}\xi}{\xi\beta}\right)^{\frac{1}{2}}$$
$$\alpha_4 = \tfrac{23}{26} + \tfrac{\sqrt{2}\beta}{156} + \tfrac{1}{156}\left(\tfrac{-3664\xi\beta+26\xi^2\beta+71786\beta+26460\sqrt{2}\xi}{\xi\beta}\right)^{\frac{1}{2}}$$

where $\xi = (136819 + 36i\sqrt{1796295})^{\frac{1}{3}}$ and $\beta = \left(\frac{13\xi^2+916\xi+35893}{\xi}\right)^{\frac{1}{2}}$.
Simple calculations show that α_i, for $i = 1, \ldots, 4$, yields positive real values for X_1 and X_2 whose approximations are given below:

i	X_1	X_2
1	0.3702	4.6215
2	0.5345	0.6682
3	1.0499	0.6338
4	1.5838	5.2195

Hence, there exist exactly four Einstein adapted metrics on M.

A.47, for $p = 2$) For the bisymmetric triple A.47, in the case $p = 2$, the non-zero solutions of the equations (3.27) and (3.28) are of the form

$$X_1 = \frac{1}{2}\alpha_i,\ X_2 = -\frac{140}{3}\alpha_i^3 + 148\alpha_i^2 - \frac{681}{5}\alpha_i + \frac{184}{5},$$

where α_i is a root of the polynomial

$$t(z) = 350z^4 - 1110z^3 + 1179z^2 - 492z + 69.$$

The polynomial t has exactly four positive real roots which are

$$\begin{aligned}
\alpha_1 &= \tfrac{111}{140} - \tfrac{\beta}{140} - \tfrac{1}{140}\left(\tfrac{2634\xi\beta - 14\xi^2\beta - 64526\beta - 35250\xi}{\xi\beta}\right)^{\frac{1}{2}}\\
\alpha_2 &= \tfrac{111}{140} - \tfrac{\beta}{140} + \tfrac{1}{140}\left(\tfrac{2634\xi\beta - 14\xi^2\beta - 64526\beta - 35250\xi}{\xi\beta}\right)^{\frac{1}{2}}\\
\alpha_3 &= \tfrac{111}{140} + \tfrac{\beta}{140} - \tfrac{1}{140}\left(\tfrac{2634\xi\beta - 14\xi^2\beta - 64526\beta - 35250\xi}{\xi\beta}\right)^{\frac{1}{2}}\\
\alpha_4 &= \tfrac{111}{140} + \tfrac{\beta}{140} + \tfrac{1}{140}\left(\tfrac{2634\xi\beta - 14\xi^2\beta - 64526\beta - 35250\xi}{\xi\beta}\right)^{\frac{1}{2}}
\end{aligned}$$

where $\xi = (290727 + 500i\sqrt{53545})^{\frac{1}{3}}$ and $\beta = \left(\frac{14\xi^2+1317\xi+64526}{\xi}\right)^{\frac{1}{2}}$.
Simple calculations show that α_i, $i = 1, \ldots, 4$, yields positive real values of X_1 and X_2 whose approximations are given below:

i	X_1	X_2
1	0.3086	7.4890
2	0.4686	0.6737
3	0.9326	0.6496
4	1.4616	8.1878

Hence, there are on M exactly four Einstein adapted metrics.

A.47, for $p = 4$) For the bisymmetric triple A.47, in the case $p = 2$, the non-zero solutions of the equations (3.27) and (3.28) are given by

$$X_1 = \alpha_i,\ X_2 = -\frac{100}{3}\alpha_i^3 + 100\alpha_i^2 - \frac{262}{3}\alpha_i + 26,$$

where α_i is a root of the polynomial

$$t(z) = 200z^4 - 600z^3 + 614z^2 - 264z + 39.$$

Since $X_1 = \alpha_i$, only positive roots of t yield positive solutions of the equations above. The polynomial t has exactly two positive roots which are indicated below:

$$\begin{aligned}\alpha_1 &= \tfrac{3\xi\beta}{4} + \tfrac{\beta^3}{60} - \tfrac{\sqrt{3}}{60}\tfrac{122\xi^2\beta^2+\xi^4\beta^2-1151\beta^2+1620\xi^3}{\xi\beta}\\ \alpha_2 &= \tfrac{3\xi\beta}{4} + \tfrac{\beta^3}{60} + \tfrac{\sqrt{3}}{60}\tfrac{122\xi^2\beta^2+\xi^4\beta^2-1151\beta^2+1620\xi^3}{\xi\beta}\end{aligned}$$

where $\xi = (109457 + 180\sqrt{416842})^{\frac{1}{6}}$ and $\beta = (14\xi^2 + 1317\xi + 64526)^{\frac{1}{4}}$.
Simple calculations show that α_i, $i = 1, \ldots, 4$, yields positive real values of X_1 and X_2 whose approximations are given below:

i	X_1	X_2
1	0.3143	7.3931
2	1.4375	8.0839

Therefore, there are on M exactly two Einstein adapted metrics.

A.47, for $p = 6$) For the bisymmetric triple A.47, in the case $p = 6$, the non-zero solutions of the equations (3.27) and (3.28) are of the form

$$X_1 = 3\alpha_i,\ X_2 = -\frac{2500}{3}z^3 + 820z^2 - 235z + 24,$$

where α_i is a root of the polynomial

$$t(z) = 1250z^4 - 1230z^3 + 415z^2 - 60z + 3.$$

This polynomial has exactly two positive roots which are

$$\begin{aligned}\alpha_1 &= \tfrac{41\times 3^{\frac{3}{4}}\xi\beta}{1500} + \tfrac{5^{\frac{2}{3}}3^{\frac{3}{4}}\beta^3}{22500} - \tfrac{5^{\frac{5}{6}}3^{\frac{3}{4}}\sqrt{6}}{22500}\tfrac{(3887\sqrt{3}5^{\frac{1}{3}}\xi^2\beta^2+125\sqrt{3}\xi^4\beta^2-20875\sqrt{3}5^{\frac{2}{3}}\beta^2+527553\times 5^{\frac{2}{3}}\xi^3)^{\frac{1}{2}}}{\xi\beta}\\ \alpha_2 &= \tfrac{41\times 3^{\frac{3}{4}}\xi\beta}{1500} + \tfrac{5^{\frac{2}{3}}3^{\frac{3}{4}}\beta^3}{22500} + \tfrac{5^{\frac{5}{6}}3^{\frac{3}{4}}\sqrt{6}}{22500}\tfrac{(3887\sqrt{3}5^{\frac{1}{3}}\xi^2\beta^2+125\sqrt{3}\xi^4\beta^2-20875\sqrt{3}5^{\frac{2}{3}}\beta^2+527553\times 5^{\frac{2}{3}}\xi^3)^{\frac{1}{2}}}{\xi\beta}\end{aligned}$$

where $\xi = (14027+18\sqrt{2}\sqrt{483323})^{\frac{1}{6}}$ and $\beta = (3887\times 5^{\frac{2}{3}}\xi^2 - 250\times 5^{\frac{1}{3}}\xi^4 + 208750)^{\frac{1}{4}}$.
Simple calculations show that α_i, $i = 1, \ldots, 2$, yields positive real values of X_1 and X_2 whose approximations are given below:

i	X_1	X_2
1	0.3163	7.3606
2	1.4292	8.0485

Hence, there are on M exactly two Einstein adapted metrics.

A.42) In this case there is no Einstein adapted metric on M. We get that, in particular, X_2 would be a root of the polynomial

$$t(z) = 9z^4 - 195z^3 + 1198z^2 - 1395z + 464,$$

but t does not admit any positive root.

□

In the classical case, similar methods as those briefly exposed in the proof above may be attempted to obtain solutions for the Einstein equations. However, as it can be read from Table 3.7 the eigenvalues depend on parameters in general which would retrieve very complicated equations. Though for the bisymmetric triples which satisfy one of the conditions $\gamma_1 = \gamma_2$ or $\gamma_2 = 1 - \gamma_1$ it is possible to classify all the Einstein adapted metrics by using Corollaries 2.14 and 2.15, respectively. For these spaces, the Einstein adapted metrics g_M whose restriction g_F is also Einstein were classified in 3.3. So it remains to obtain all the other possible Einstein adapted metrics.

Theorem 3.5. *Let $F \to M \to N$ be a bisymmetric fibration of Type II such that $\gamma_2 = \gamma_1$ or $\gamma_2 = 1 - \gamma_1$. If there exists on M an Einstein adapted metric such that g_F is not Einstein, then the corresponding bisymmetric triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ is one of the triples in Table 3.11.*

Proof: In the only case when $\gamma_2 = 1 - \gamma_1$, A.3 for $p = 2l$ and $n = 2(l + s)$, we have $\gamma_1 = \frac{l}{l+s}$, $\gamma_2 = \frac{s}{l+s}$, $b_1 = \frac{l}{4(l+s)} = \frac{\gamma_1}{4}$ and $b_2 = \frac{s}{4(l+s)} = \frac{1-\gamma_1}{4}$. The required metric should be given by Corollary 2.15 (ii). Simple calculation show that $D(\gamma_1) = -\frac{1}{2}\gamma_1(1 - \gamma_1) < 0$, for every l, s, since $0 < \gamma_1 < 1$. Hence in this case there are no other Einstein adapted metrics besides those found previously.

The cases such that $\gamma_2 = \gamma_1$ are those listed in the proof of Theorem 3.2. In this case, there exists an Einstein adapted metric such that g_F is not Einstein if and only if $D(\gamma_1) \geq 0$, where

$$D(\gamma_1) = 4r^2(1 - \gamma_1) - 2\gamma_1(2b_2 + 1 - \gamma_1)(2b_1 + 1 - \gamma_1),$$

according to Corollary 2.14 (ii).

In the cases A.3 for $s = l = 2p$, $n = 4l$, A.15 for $n = 2p$, A.23, for $n = 2p$, as in the proof of 3.10 we have $b_1 = b_2 = \frac{\gamma}{4}$. Hence, we simplify the expression for $D(\gamma_1)$ given above as

$$D(\gamma_1) = \frac{1}{2}(-\gamma_1^3 + 4\gamma_1^2 - 6\gamma_1 + 2) \tag{3.29}$$

For A.3, $s = l = 2p$, $n = 4l$, we have $\gamma_1 = \frac{1}{2}$ and $D(\frac{1}{2}) < 0$; for A.15, $n = 2p$, $p \geq 2$, we have $\gamma_1 = \frac{p-1}{2p-1}$ and $D(\gamma_1) \geq 0$ only for $p = 2, \ldots, 6$; for A.23, $n = 2p$, $\gamma = \frac{p+1}{2p+1}$ and $D < 0$, for every $p \geq 1$.

For the case A.12, with $s = l = 2p$, $n = 4l$, $l \geq 2$, as in the proof of 3.10, we have $\gamma_1 = \frac{2l-1}{4l-1}$ and $b_1 = b_2 = \frac{l}{2(4l-1)} = \frac{1-\gamma_1}{4}$. For this case, we have

$$D(\gamma_1) = -\frac{1}{2}(\gamma_1 - 1)(3\gamma_1 - 1)(3\gamma_1 - 2).$$

Since $\gamma \in (\frac{3}{7}, \frac{1}{2})$, we have $D(\gamma_1) < 0$, for every γ and thus there is no positive solution.

For A.16, where $p = 2l$, $n = 4l$, we have $\gamma_1 = \frac{2l-1}{4l-1}$ and $b_1 = \frac{l}{2(4l-1)} = \frac{1-\gamma_1}{4}$, $b_2 = \frac{2l-1}{4(4l-1)} = \frac{\gamma_1}{4}$. We rewrite $D(\gamma_1)$ as follows:

$$D(\gamma_1) = \frac{1}{2}(1 - \gamma_1)(3\gamma_1^2 - 6\gamma_1 + 2).$$

Since $\gamma_1 = \frac{2l-1}{4l-1} \in (\frac{1}{3}, \frac{1}{2})$, simple calculations show that $3\gamma_1^2 - 6\gamma_1 + 2 \geq 0$ only for $l = 1$. So only for $l = 1$ exists a metric with the desired properties.

For A.20, with $s = l = 2p$, $n = 4l$, we have $\gamma_1 = \frac{2l+1}{4l+1}$ and $b_1 = b_2 = \frac{l}{4(4l+1)} = \frac{1-\gamma_1}{8}$. Hence

$$D(\gamma_1) = \frac{1}{8}(1 - \gamma_1)(25\gamma_1^2 - 25\gamma_1 + 8).$$

As $25\gamma_1^2 - 25\gamma_1 + 8 > 0$, for every γ_1, there exists a metric with the required properties for every $l \geq 1$.

In the case A.24, for $p = 2l$, $n = 4l$, we have $\gamma_1 = \frac{2l+1}{4l+1}$ and $b_1 = \frac{l}{4(4l+1)} = \frac{1-\gamma_1}{8}$, $b_2 = \frac{2l+1}{4(4l+1)} = \frac{\gamma_1}{4}$. Thus

$$D(\gamma_1) = \frac{1}{4}(1 - \gamma_1)(5\gamma_1^2 - 10\gamma_1 + 4).$$

Since $\gamma_1 \in (\frac{1}{2}, \frac{3}{5})$, we conclude that $5\gamma_1^2 - 10\gamma_1 + 4 \geq 0$, for every $l \geq 3$. Hence, there exists an adapted Einstein metric for every $l \geq 3$.

□

Remark 3.4. *For the triples in Lemma 3.4, where $C_{\mathfrak{p}_i}$ is scalar on $\mathfrak{n}$, such that $D(\gamma_1) < 0$, we can still consider the non-real complex solutions X_1, X_2 which gives rise to a non binormal Einstein adapted metric on M with non-real complex coefficients. The spaces in these conditions are the cases A.3 for solutions of the* *form* $X_2 = \frac{1}{2X_1}$, *and for solutions of the form* $X_2 = \frac{\gamma X_1}{1-\gamma}$ *we have the cases A.3,* $s = p = 2l$, $n = 4l$, *for every* l, *A.15,* $n = 2p$, *for* $p \geq 5$, *A.23,* $n = 2p$, *for every* p, *A.12, with* $s = p = 2l$, $n = 4l$, *for* $l \geq 2$, *A.16,* $p = 2l$, $n = 4l$, *for* $l \geq 2$, *A.24,* $p = 2l$, $n = 4l$, *for* $l = 1, 2$.

3.5 Application to 4-symmetric Spaces

A homogeneous space G/L is said to be a 4-symmetric space if there exists $\sigma \in Aut(G)$ such that

$$(G_\sigma)_0 \subset L \subset G_\sigma$$

and σ has order 4. Compact simply connected irreducible 4-symmetric spaces have been classified by J.A.Jimenez in [20] following the previous work of V.Kač (see e.g. [16], Chap.X), J.A Wolf and A.Gray [48]. It is shown in [20] that any compact simply connected irreducible 4-symmetric space is the total space of a fiber bundle whose fiber and base space are symmetric spaces and the base is an isotropy irreducible space of maximal rank. These spaces are fully described in Tables III, IV and V in [20]. Hence, for each compact simply connected irreducible 4-symmetric space M there is a bisymmetric fibration $F \to M \to N$ of maximal rank whose base space N is isotropy irreducible. The bisymmetric triples $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ corresponding to 4-symmetric spaces of maximal rank must be some of Tables 3.4, 3.5, 3.6 and 3.7. Hence, a simple comparison between the Tables in this chapter and the classification in [20] allow us to easily conclude about the existence of Einstein metrics on 4-symmetric spaces.
From Theorem 3.1 and Tables 3.8 and 3.9 we conclude the following:

Corollary 3.2. *Let* $M = G/L$ *be a compact simply connected irreducible* 4*-symmetric spaces of Type I and* $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ *a bisymmetric triple corresponding to* M *such that* $L \subset K$. *If* M *admits an Einstein adapted metric, then* $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ *is one of the triples listed below.*
(i) In the exceptional case:

$$(\mathfrak{f}_4, \mathfrak{so}_9, \mathfrak{so}_7 \oplus \mathbb{R}),$$
$$(\mathfrak{f}_4, \mathfrak{sp}_3 \oplus \mathfrak{su}_2, \mathfrak{sp}_3 \oplus \mathbb{R}),$$
$$(\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2, \mathfrak{su}_2 \oplus \mathbb{R}), (\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2, \mathbb{R} \oplus \mathfrak{su}_2),$$
$$(\mathfrak{e}_8, \mathfrak{so}_{16}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{16-2p}),\ p = 1, 3,$$
$$(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{so}_{12} \oplus \mathbb{R}),$$
$$(\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathbb{R}, \mathfrak{so}_8 \oplus \mathbb{R} \oplus \mathbb{R}),$$
$$(\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2, \mathfrak{su}_6 \oplus \mathbb{R}),$$
$$(\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2, \mathfrak{su}_5 \oplus \mathbb{R} \oplus \mathfrak{su}_2),$$

(ii) in the classical case:

$$\begin{aligned}&(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2p+1} \oplus \mathfrak{u}_{n-p}),\\ &(\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{u}_p \oplus \mathfrak{so}_{2(n-p)}),\\ &(\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{u}_p \oplus \mathfrak{sp}_{n-p}).\end{aligned}$$

From Theorem 3.3 and Tables 3.10 and 3.11 we obtain the following results for Type II.

Corollary 3.3. *Let $M = G/L$ be a compact simply connected irreducible 4-symmetric spaces of Type II and $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ a bisymmetric triple corresponding to M such that $L \subset K$. If M admits an Einstein adapted metric g_M such that g_F is also Einstein, then $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ is either*

$$(\mathfrak{so}_{8l}, \mathfrak{so}_{4l} \oplus \mathfrak{so}_{4l}, \mathfrak{so}_{2l} \oplus \mathfrak{so}_{2l} \oplus \mathfrak{so}_{2l} \oplus \mathfrak{so}_{2l})$$

or

$$(\mathfrak{su}_{2(l+s)}, \mathfrak{su}_{2l} \oplus \mathfrak{su}_{2s} \oplus \mathbb{R}, \mathfrak{su}_l \oplus \mathfrak{su}_l \oplus \mathfrak{su}_s \oplus \mathfrak{su}_s \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}).$$

The Einstein metric is binormal in the first case and in the second for $l = s$.

Finally, the result below follows from Theorem 3.4 and Table 3.6.

Corollary 3.4. *Let $M = G/L$ be a compact simply connected irreducible 4-symmetric spaces of Type II, such that G is an exceptional Lie group, and $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ a bisymmetric triple corresponding to M such that $L \subset K$. If M admits an Einstein adapted metric, then $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ is one of the following three cases:*

$$\begin{aligned}&(\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2, \mathfrak{su}_5 \oplus \mathbb{R} \oplus \mathbb{R})\\ &(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{so}_{10} \oplus \mathbb{R} \oplus \mathbb{R})\\ &(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{so}_6 \oplus \mathfrak{so}_6 \oplus \mathbb{R}).\end{aligned}$$

There are 4 Einstein adapted metrics for each of the first two cases and 2 for the last case. None of these metrics is binormal or such that the restriction to the fiber is Einstein.

Remark 3.5. *We observe that for the 4-symmetric spaces corresponding to the first two cases the base space considered by Jimenez is different from the one indicated above. In this two cases, the bisymmetric triples considered in [20] are*

$$\begin{aligned}&(\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathbb{R}, \mathfrak{su}_5 \oplus \mathbb{R} \oplus \mathbb{R})\\ &(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R}, \mathfrak{so}_{10} \oplus \mathbb{R} \oplus \mathbb{R}).\end{aligned}$$

For these triples there is no Einstein adapted metric since the Casimir operator of $\mathfrak{p}$ is not scalar.

3.6 Tables

Table 3.1: Dual Coxeter Numbers

Coxeter group	Dual Coxeter number
A_n	$n+1$
B_n	$2n-1$
C_n	$n+1$
D_n	$2n-2$
E_6	12
E_7	18
E_8	30
F_4	9
G_2	4

Table 3.2: Symmetric pairs of compact type of maximal rank - Exceptional Spaces

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{g}$	$\mathfrak{k}$
$\mathfrak{f}_4$	$\mathfrak{sp}_3 \oplus \mathfrak{su}_2$	$\mathfrak{e}_7$	$\mathfrak{su}_8$
$\mathfrak{f}_4$	$\mathfrak{so}_9$	$\mathfrak{e}_7$	$\mathfrak{e}_6 \oplus \mathbb{R}$
$\mathfrak{g}_2$	$\mathfrak{su}_2 \oplus \mathfrak{su}_2$	$\mathfrak{e}_7$	$\mathfrak{so}_{12} \oplus \mathfrak{su}_2$
$\mathfrak{e}_6$	$\mathfrak{so}_{10} \oplus \mathbb{R}$	$\mathfrak{e}_8$	$\mathfrak{so}_{16}$
$\mathfrak{e}_6$	$\mathfrak{su}_6 \oplus \mathfrak{su}_2$	$\mathfrak{e}_8$	$\mathfrak{e}_7 \oplus \mathfrak{su}_2$

Table 3.3: Symmetric pairs of compact type of maximal rank - Classical Spaces

$\mathfrak{g}$	$\mathfrak{k}$
$\mathfrak{so}_{2n}$	$\mathfrak{u}_n$
$\mathfrak{so}_n$	$\mathfrak{so}_{2p} \oplus \mathfrak{so}_{n-2p}$
$\mathfrak{sp}_n$	$\mathfrak{u}_n$
$\mathfrak{sp}_n$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}$
$\mathfrak{su}_n$	$\mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}$

Table 3.4: Bisymmetric triples of type I and their eigenvalues - Exceptional spaces

Bisymmetric triple	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{l}$	γ	b^{ϕ}
$A.25$	$\mathfrak{f}_4$	$\mathfrak{so}_9$	$\mathfrak{so}_p \oplus \mathfrak{so}_{9-p}$, $p = 1, 3, 5, 7$	$\frac{7}{9}$	$\frac{p(9-p)}{72}$
$A.26$	$\mathfrak{f}_4$	$\mathfrak{sp}_3 \oplus \mathfrak{su}_2$	$\mathfrak{sp}_3 \oplus \mathbb{R}$	$\frac{2}{9}$	$\frac{1}{18}$
$A.27$			$\mathfrak{u}_3 \oplus \mathfrak{su}_2$	$\frac{4}{9}$	$\frac{1}{4}, \frac{2}{9}$
$A.28$			$\mathfrak{sp}_2 \oplus \mathfrak{su}_2 \oplus \mathfrak{su}_2$	$\frac{4}{9}$	$\frac{1}{9}, \frac{1}{18}$
$A.31$	$\mathfrak{g}_2$	$\mathfrak{su}_2 \oplus \mathfrak{su}_2$	$\mathbb{R} \oplus \mathfrak{su}_2$	$\frac{1}{2}$	$\frac{1}{8}$
$A.32$			$\mathfrak{su}_2 \oplus \mathbb{R}$	$\frac{1}{6}$	$\frac{1}{6}$
$A.34$	$\mathfrak{e}_8$	$\mathfrak{so}_{16}$	$\mathfrak{so}_{2p} \oplus \mathfrak{so}_{16-2p}$, $p = 1, \ldots, 4$	$\frac{1}{5}$	$\frac{p(8-p)}{60}$
$A.35$			$\mathfrak{u}_8$	$\frac{1}{5}$	$\frac{4}{15}, \frac{3}{15}, \frac{7}{15}$
$A.36$	$\mathfrak{e}_8$	$\mathfrak{e}_7 \oplus \mathfrak{su}_2$	$\mathfrak{e}_7 \oplus \mathbb{R}$	$\frac{1}{15}$	$\frac{1}{60}$
$A.37$			$\mathfrak{e}_6 \oplus \mathbb{R} \oplus \mathfrak{su}_2$	$\frac{3}{5}$	$\frac{11}{60}, \frac{9}{20}$
$A.39$			$\mathfrak{so}_{12} \oplus \mathfrak{su}_2 \oplus \mathfrak{su}_2$	$\frac{3}{5}$	$\frac{4}{15}, \frac{1}{5}$
$A.41$			$\mathfrak{su}_8 \oplus \mathfrak{su}_2$	$\frac{3}{5}$	$\frac{1}{4}$
$A.43$	$\mathfrak{e}_7$	$\mathfrak{so}_{12} \oplus \mathfrak{su}_2$	$\mathfrak{so}_{12} \oplus \mathbb{R}$	$\frac{1}{9}$	$\frac{1}{36}$
$A.44$			$\mathfrak{u}_6 \oplus \mathfrak{su}_2$	$\frac{5}{9}$	$\frac{1}{6}, \frac{5}{18}$
$A.46$			$\mathfrak{so}_p \oplus \mathfrak{so}_{12-p} \oplus \mathfrak{su}_2$, $p = 2, 4, 6$	$\frac{5}{9}$	$\frac{p(12-p)}{144}$
$A.48$	$\mathfrak{e}_7$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{so}_{10} \oplus \mathbb{R} \oplus \mathbb{R}$	$\frac{2}{3}$	$\frac{2}{9}, \frac{1}{6}, \frac{4}{9}$
$A.49$			$\mathfrak{su}_6 \oplus \mathfrak{su}_2 \oplus \mathbb{R}$	$\frac{2}{3}$	$\frac{5}{18}, \frac{2}{9}$
$A.50$	$\mathfrak{e}_7$	$\mathfrak{su}_8$	$\mathfrak{su}_p \oplus \mathfrak{su}_{8-p} \oplus \mathbb{R}$, $1 \leq p \leq 4$	$\frac{4}{9}$	$\frac{1}{9}$, $p = 1$ $\frac{2}{9}, \frac{1}{6}$, $p = 2$ $\frac{2}{9}, \frac{1}{3}$, $p = 3$ $\frac{2}{9}, \frac{4}{9}, \frac{11}{36}$, $p = 4$
$A.51$	$\mathfrak{e}_6$	$\mathfrak{so}_{10} \oplus \mathbb{R}$	$\mathfrak{u}_5 \oplus \mathbb{R}$	$\frac{2}{3}$	$\frac{5}{12}, \frac{1}{6}, \frac{1}{4}$
$A.52$			$\mathfrak{so}_p \oplus \mathfrak{so}_{10-p} \oplus \mathbb{R}$, $p = 2, 4$	$\frac{2}{3}$	$\frac{p(10-p)}{96}$
$A.53$	$\mathfrak{e}_6$	$\mathfrak{su}_6 \oplus \mathfrak{su}_2$	$\mathfrak{su}_6 \oplus \mathbb{R}$	$\frac{1}{6}$	$\frac{1}{24}$
$A.54$			$\mathfrak{su}_p \oplus \mathfrak{su}_{6-p} \oplus \mathbb{R} \oplus \mathfrak{su}_2$	$\frac{1}{2}$	$\frac{p+2}{24}, \frac{p}{8}$

Table 3.5: Bisymmetric triples of type I and their eigenvalues - Classical spaces

Bisymmetric triple	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{l}$	γ	b^{ϕ}
$A.1$	$\mathfrak{su}_n$	$\mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}$	$\mathfrak{su}_l \oplus \mathfrak{su}_{p-l} \oplus \mathbb{R} \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}$	$\frac{p}{n}$	$\frac{p-l}{2n}$, $\frac{l}{2n}$
$A.4$	$\mathfrak{so}_{2n+1}$	$\mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}$, $p = 0, \ldots, n-1$	$\mathfrak{so}_{2l+1} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2(n-p)}$	$\frac{2p-1}{2n-1}$	$\frac{p-l}{2n-1}$, $\frac{4l+1}{4(2n-1)}$
$A.5$			$\mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2s} \oplus \mathfrak{so}_{2(n-p-s)}$	$\frac{2(n-p-1)}{2n-1}$	$\frac{n-p-s}{2n-1}$, $\frac{s}{2n-1}$
$A.6$			$\mathfrak{so}_{2p+1} \oplus \mathfrak{u}_{n-p}$	$\frac{2(n-p-1)}{2n-1}$	$\frac{n-p-1}{2(2n-1)}$
$A.9$	$\mathfrak{so}_{2n}$	$\mathfrak{u}_n$	$\mathfrak{u}_p \oplus \mathfrak{u}_{n-p}$, $p = 1, \ldots, n-1$	$\frac{n}{2(n-1)}$	$\frac{n-p}{2(n-1)}$, $\frac{p}{2(n-1)}$, $\frac{n-2}{4(n-1)}$
$A.10$	$\mathfrak{so}_{2n}$	$\mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}$, $p = 1, \ldots, \lfloor \frac{n}{2} \rfloor$	$\mathfrak{so}_{2l} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2(n-p)}$	$\frac{p-1}{n-1}$	$\frac{p-l}{2(n-1)}$, $\frac{l}{2(n-1)}$
$A.13$			$\mathfrak{u}_p \oplus \mathfrak{so}_{2(n-p)}$	$\frac{p-1}{n-1}$	$\frac{p-1}{4(n-1)}$
$A.17$	$\mathfrak{sp}_n$	$\mathfrak{u}_n$	$\mathfrak{u}_p \oplus \mathfrak{u}_{n-p}$, $p = 1, \ldots, n-1$	$\frac{n}{2(n+1)}$	$\frac{n-p}{2(n+1)}$, $\frac{n-p}{n+1}$, $\frac{p}{2(n+1)}$, $\frac{p}{n+1}$, $\frac{n+2}{2(n+1)}$
$A.18$	$\mathfrak{sp}_n$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}$	$\mathfrak{sp}_l \oplus \mathfrak{sp}_{p-l} \oplus \mathfrak{sp}_{n-p}$	$\frac{p+1}{n+1}$	$\frac{p-l}{4(n+1)}$, $\frac{l}{4(n+1)}$
$A.21$			$\mathfrak{u}_p \oplus \mathfrak{sp}_{n-p}$	$\frac{p+1}{n+1}$	$\frac{p+1}{4(n+1)}$

Table 3.6: Bisymmetric triples of type II and their eigenvalues - Exceptional spaces

Bisymmetric triple	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{l}$	γ_1	γ_2	b_1^ϕ	b_2^ϕ
$A.29$	$\mathfrak{f}_4$	$\mathfrak{sp}_3 \oplus \mathfrak{su}_2$	$\mathfrak{u}_3 \oplus \mathbb{R}$	$\frac{4}{9}$	$\frac{2}{9}$	$\left(\frac{1}{4}, \frac{2}{9}\right)$	$\frac{1}{18}$
$A.30$			$\mathfrak{su}_2 \oplus \mathfrak{sp}_2 \oplus \mathbb{R}$	$\frac{4}{9}$	$\frac{2}{9}$	$\left(\frac{1}{9}, \frac{1}{18}\right)$	$\frac{1}{18}$
$A.33$	$\mathfrak{g}_2$	$\mathfrak{su}_2 \oplus \mathfrak{su}_2$	$\mathbb{R} \oplus \mathbb{R}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{6}$
$A.38$	$\mathfrak{e}_8$	$\mathfrak{e}_7 \oplus \mathfrak{su}_2$	$\mathfrak{e}_6 \oplus \mathbb{R} \oplus \mathbb{R}$	$\frac{3}{5}$	$\frac{1}{15}$	$\left(\frac{11}{60}, \frac{11}{60}, \frac{9}{20}\right)$	$\frac{1}{60}$
$A.40$			$\mathfrak{so}_{12} \oplus \mathfrak{su}_2 \oplus \mathbb{R}$	$\frac{3}{5}$	$\frac{1}{15}$	$\left(\frac{4}{15}, \frac{1}{5}\right)$	$\frac{1}{60}$
$A.42$			$\mathfrak{su}_8 \oplus \mathbb{R}$	$\frac{3}{5}$	$\frac{1}{15}$	$\frac{1}{4}$	$\frac{1}{60}$
$A.45$	$\mathfrak{e}_7$	$\mathfrak{so}_{12} \oplus \mathfrak{su}_2$	$\mathfrak{u}_6 \oplus \mathbb{R}$	$\frac{5}{9}$	$\frac{1}{9}$	$\left(\frac{5}{18}, \frac{1}{6}, \frac{5}{18}\right)$	$\frac{1}{36}$
$A.47$			$\mathfrak{so}_p \oplus \mathfrak{so}_{12-p} \oplus \mathbb{R}$, p even	$\frac{5}{9}$	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{p(12-p)}{144}$
$A.55$	$\mathfrak{e}_6$	$\mathfrak{su}_6 \oplus \mathfrak{su}_2$	$\mathfrak{su}_p \oplus \mathfrak{su}_{6-p} \oplus \mathbb{R} \oplus \mathbb{R}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{p+2}{24}, \frac{p}{8}$

Table 3.7: Bisymmetric triples of type II and their eigenvalues - Classical spaces

Bisymmetric triple	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{l}$	γ_1	γ_2	b_1^ϕ	b_2^ϕ
A.3	$\mathfrak{su}_n$	$\mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}$	$\mathfrak{su}_l \oplus \mathfrak{su}_{p-l} \oplus \mathfrak{su}_s \oplus \mathfrak{su}_{n-p-s} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	$\frac{p}{n}$	$\frac{n-p}{n}$	$\left(\frac{p-l}{2n}, \frac{l}{2n}\right)$	$\left(\frac{n-p-s}{2n}, \frac{s}{2n}\right)$
A.8	$\mathfrak{so}_{2n+1}$	$\mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}$	$\mathfrak{so}_{2l+1} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2s} \oplus \mathfrak{so}_{2(n-p-s)}$	$\frac{2p-1}{2n-1}$	$\frac{2(n-p-1)}{2n-1}$	$\left(\frac{p-l}{2n-1}, \frac{4l+1}{4(2n-1)}\right)$	$\left(\frac{n-p-s}{2n-1}, \frac{s}{2n-1}\right)$
A.7			$\mathfrak{so}_{2l+1} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{u}_{n-p}$	$\frac{2p-1}{2n-1}$	$\frac{2(n-p-1)}{2n-1}$	$\left(\frac{p-l}{2n-1}, \frac{4l+1}{4(2n-1)}\right)$	$\frac{n-p-1}{2(2n-1)}$
A.12	$\mathfrak{so}_{2n}$	$\mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}$	$\mathfrak{so}_{2l} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2s} \oplus \mathfrak{so}_{2(n-p-s)}$	$\frac{p-1}{n-1}$	$\frac{n-p-1}{n-1}$	$\left(\frac{p-l}{2(n-1)}, \frac{l}{2(n-1)}\right)$	$\left(\frac{n-p-s}{2(n-1)}, \frac{s}{2(n-1)}\right)$
A.15			$\mathfrak{u}_p \oplus \mathfrak{u}_{n-p}$	$\frac{p-1}{n-1}$	$\frac{n-p-1}{n-1}$	$\frac{p-1}{4(n-1)}$	$\frac{n-p-1}{4(n-1)}$
A.16			$\mathfrak{so}_{2l} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{u}_{n-p}$	$\frac{p-1}{n-1}$	$\frac{n-p-1}{n-1}$	$\left(\frac{p-l}{2(n-1)}, \frac{l}{2(n-1)}\right)$	$\frac{n-p-1}{4(n-1)}$
A.20	$\mathfrak{sp}_n$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}$	$\mathfrak{sp}_l \oplus \mathfrak{sp}_{p-l} \oplus \mathfrak{sp}_s \oplus \mathfrak{sp}_{n-p-s}$	$\frac{p+1}{n+1}$	$\frac{n-p+1}{n+1}$	$\left(\frac{p-l}{4(n+1)}, \frac{l}{4(n+1)}\right)$	$\left(\frac{n-p-s}{4(n+1)}, \frac{s}{4(n+1)}\right)$
A.23			$\mathfrak{u}_p \oplus \mathfrak{u}_{n-p}$	$\frac{p+1}{n+1}$	$\frac{n-p+1}{n+1}$	$\frac{p+1}{4(n+1)}$	$\frac{n-p+1}{4(n+1)}$
A.24			$\mathfrak{sp}_l \oplus \mathfrak{sp}_{p-l} \oplus \mathfrak{u}_{n-p}$	$\frac{p+1}{n+1}$	$\frac{n-p+1}{n+1}$	$\left(\frac{p-l}{4(n+1)}, \frac{l}{4(n+1)}\right)$	$\frac{n-p+1}{4(n+1)}$

Table 3.8: Einstein Bisymmetric triples of type I - Exceptional spaces

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{l}$	X
$\mathfrak{f}_4$	$\mathfrak{sp}_3 \oplus \mathfrak{su}_2$	$\mathfrak{sp}_3 \oplus \mathbb{R}$	$\frac{1}{2}$, 4
$\mathfrak{f}_4$	$\mathfrak{so}_9$	$\mathfrak{so}_8$	1, $\frac{2}{7}$
		$\mathfrak{so}_7 \oplus \mathbb{R}$	$\frac{9\pm\sqrt{8}}{14}$
$\mathfrak{g}_2$	$\mathfrak{su}_2 \oplus \mathfrak{su}_2$	$\mathbb{R} \oplus \mathfrak{su}_2$	$\frac{1}{2}$, $\frac{3}{2}$
		$\mathfrak{su}_2 \oplus \mathbb{R}$	$\frac{6\pm\sqrt{22}}{2}$
$\mathfrak{e}_6$	$\mathfrak{so}_{10} \oplus \mathbb{R}$	$\mathfrak{so}_8 \oplus \mathbb{R} \oplus \mathbb{R}$	1, $\frac{1}{2}$
$\mathfrak{e}_6$	$\mathfrak{su}_6 \oplus \mathfrak{su}_2$	$\mathbb{R} \oplus \mathfrak{su}_5 \oplus \mathfrak{su}_2$	$\frac{1}{2}$, $\frac{3}{2}$
		$\mathfrak{su}_6 \oplus \mathbb{R}$	$\frac{1}{2}$, $\frac{11}{2}$
$\mathfrak{e}_7$	$\mathfrak{so}_{12} \oplus \mathfrak{su}_2$	$\mathfrak{so}_{12} \oplus \mathbb{R}$	$\frac{17}{2}$, $\frac{1}{2}$
		$\mathbb{R} \oplus \mathfrak{so}_{10} \oplus \mathfrak{su}_2$	$\frac{1}{2}$, $\frac{13}{10}$
		$\mathfrak{so}_4 \oplus \mathfrak{so}_8 \oplus \mathfrak{su}_2$	1, $\frac{4}{5}$
$\mathfrak{e}_7$	$\mathfrak{su}_8$	$\mathfrak{su}_7 \oplus \mathbb{R}$	$\frac{1}{2}$, $\frac{7}{4}$
$\mathfrak{e}_8$	$\mathfrak{e}_7 \oplus \mathfrak{su}_2$	$\mathfrak{e}_7 \oplus \mathbb{R}$	$\frac{1}{2}$, $\frac{29}{2}$
$\mathfrak{e}_8$	$\mathfrak{so}_{16}$	$\mathfrak{so}_{2p} \oplus \mathfrak{so}_{16-2p}$	$\frac{15\pm\sqrt{7p^2-56p+113}}{14}$

Table 3.9: Einstein Bisymmetric triples of type I - Classical spaces

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{l}$	X
$\mathfrak{so}_{2n}$	$\mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}$	$\mathfrak{so}_p \oplus \mathfrak{so}_p \oplus \mathfrak{so}_{2(n-p)}$, $p\, even$	$\frac{n-1\pm\sqrt{p^2-(2n+1)p+n^2+1}}{2(p-1)}$
		$\mathfrak{u}_p \oplus \mathfrak{so}_{2(n-p)}$	$\frac{1}{2},\ \frac{n}{p-1} - \frac{1}{2}$
$\mathfrak{so}_{2n+1}$	$\mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}$	$\mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{n-p} \oplus \mathfrak{so}_{n-p}$, $n-p\, even$	$\frac{2n-1\pm\sqrt{4p^2+8p-4n+5}}{4(n-p-1)}$
		$\mathfrak{so}_{2p+1} \oplus \mathfrak{u}_{n-p}$	$\frac{1}{2},\ \frac{n+p}{2(n-p-1)}$
$\mathfrak{sp}_n$	$\mathfrak{sp}_{2l} \oplus \mathfrak{sp}_{n-2l}$	$\mathfrak{sp}_l \oplus \mathfrak{sp}_l \oplus \mathfrak{sp}_{n-2l}$	$\frac{n+1\pm\sqrt{6l^2+(3-4n)l+n^2+1}}{2(2l+1)}$
		$\mathfrak{u}_p \oplus \mathfrak{sp}_{n-p}$	$\frac{1}{2},\ \frac{1}{2} + \frac{n-p}{p+1}$
$\mathfrak{su}_n$	$\mathfrak{su}_{2l} \oplus \mathfrak{su}_{n-2l} \oplus \mathbb{R}$	$\mathfrak{su}_l \oplus \mathfrak{su}_l \oplus \mathbb{R} \oplus \mathfrak{su}_{n-2l} \oplus \mathbb{R}$	$\frac{1}{2},\ \frac{n}{2l} - \frac{1}{2}$

Table 3.10: Bisymmetric triples of type II with Einstein metric such that g_F is also Einstein

g_M binormal

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{l}$	X
$\mathfrak{su}_{4l}$	$\mathfrak{su}_{2l} \oplus \mathfrak{su}_{2l} \oplus \mathbb{R}$	$\mathfrak{su}_l \oplus \mathfrak{su}_l \oplus \oplus \mathfrak{su}_l \oplus \mathfrak{su}_l \oplus \mathbb{R}^3$	1
$\mathfrak{so}_{8l}$	$\mathfrak{so}_{4l} \oplus \mathfrak{so}_{4l}$	$\mathfrak{so}_{2l} \oplus \mathfrak{so}_{2l} \oplus \mathfrak{so}_{2l} \oplus \mathfrak{so}_{2l}$	$1, \frac{2l}{2l-1}$
$\mathfrak{so}_{8l}$	$\mathfrak{so}_{4l} \oplus \mathfrak{so}_{4l}$	$\mathfrak{so}_{2l} \oplus \mathfrak{so}_{2l} \oplus \mathfrak{u}_{2l}$	$\frac{4l-1\pm\sqrt{2l}}{2(2l-1)}$
$\mathfrak{so}_{4l}$	$\mathfrak{so}_{2l} \oplus \mathfrak{so}_{2l}$	$\mathfrak{u}_l \oplus \mathfrak{u}_l$, $l \geq 2$	$\frac{2l-1\pm\sqrt{2l-1}}{2(l-1)}$
$\mathfrak{sp}_{4l}$	$\mathfrak{sp}_{2l} \oplus \mathfrak{sp}_{2l}$	$\mathfrak{sp}_l \oplus \mathfrak{sp}_l \oplus \mathfrak{sp}_l \oplus \mathfrak{sp}_l$	$\frac{4l+1\pm\sqrt{4l^2+2l+1}}{2(2l+1)}$
$\mathfrak{sp}_{4l}$	$\mathfrak{sp}_{2l} \oplus \mathfrak{sp}_{2l}$	$\mathfrak{sp}_l \oplus \mathfrak{sp}_l \oplus \mathfrak{u}_{2l}$	$\frac{4l+1\pm\sqrt{l(2l-1)}}{2(2l+1)}$

g_M non-binormal

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{l}$	X_1	X_2
$\mathfrak{su}_{2(l+s)}$	$\mathfrak{su}_{2l} \oplus \mathfrak{su}_{2s} \oplus \mathbb{R}$	$\mathfrak{su}_l \oplus \mathfrak{su}_l \oplus \mathfrak{su}_s \oplus \mathfrak{su}_s \oplus \mathbb{R}^3$	$\frac{l+s}{2l}$	$\frac{l+s}{2s}$

Table 3.11: All other Einstein adapted metrics for the bisymmetric triples of Type II which admit an EAM g_M such that g_F is Einstein

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{l}$	X_1	X_2
$\mathfrak{so}_{4l}$	$\mathfrak{so}_{2l} \oplus \mathfrak{so}_{2l}$	$\mathfrak{u}_l \oplus \mathfrak{u}_l$, $l = 2, \dots, 6$	$\frac{2l(l-1) \pm \sqrt{(-l^4+7l^3-5l^2+l)/2}}{2(l-1)(3l-1)}$	$\frac{l}{2(l-1)} \cdot \frac{1}{X_1}$
$\mathfrak{so}_8$	$\mathfrak{so}_4 \oplus \mathfrak{so}_4$	$\mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{u}_2$	$\frac{4 \pm \sqrt{6}}{5}$	$\frac{1}{X_1}$
$\mathfrak{sp}_{4l}$	$\mathfrak{sp}_{2l} \oplus \mathfrak{sp}_{2l}$	$\mathfrak{sp}_l \oplus \mathfrak{sp}_l \oplus \mathfrak{sp}_l \oplus \mathfrak{sp}_l$, $l \geq 1$	$\frac{4l+1 \pm \sqrt{14l^2+7l+4}}{5(2l+1)}$	$\frac{l}{2l+1} \cdot \frac{1}{X_1}$
$\mathfrak{sp}_{4l}$	$\mathfrak{sp}_{2l} \oplus \mathfrak{sp}_{2l}$	$\mathfrak{sp}_l \oplus \mathfrak{sp}_l \oplus \mathfrak{u}_{2l}$, $l \geq 3$	$\frac{2(4l+1) \pm \sqrt{4l^2-8l-1}}{5(2l+1)}$	$\frac{l}{2l+1} \cdot \frac{1}{X_1}$

Table 3.12: Einstein bisymmetric fibrations of Type II with $\mathfrak{g}$ exceptional

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{l}$	$Number\, of\, Einstein\, adapted\, metrics$
$\mathfrak{g}_2$	$\mathfrak{su}_2 \oplus \mathfrak{su}_2$	$\mathbb{R} \oplus \mathbb{R}$	2
$\mathfrak{e}_6$	$\mathfrak{su}_6 \oplus \mathfrak{su}_2$	$\mathfrak{su}_5 \oplus \mathbb{R} \oplus \mathbb{R}$	4
$\mathfrak{e}_7$	$\mathfrak{so}_{12} \oplus \mathfrak{su}_2$	$\mathbb{R} \oplus \mathfrak{so}_{10} \oplus \mathbb{R}$	4
$\mathfrak{e}_7$	$\mathfrak{so}_{12} \oplus \mathfrak{su}_2$	$\mathfrak{so}_4 \oplus \mathfrak{so}_8 \oplus \mathbb{R}$	2
$\mathfrak{e}_7$	$\mathfrak{so}_{12} \oplus \mathfrak{su}_2$	$\mathfrak{so}_6 \oplus \mathfrak{so}_6 \oplus \mathbb{R}$	2

CHAPTER 4

In this Chapter we consider a fibration

$$\frac{\triangle^p G_0 \times \triangle^q G_0}{\triangle^n G_0} \to \frac{G_0^n}{\triangle^n G_0} \to \frac{G_0^p}{\triangle^p G_0} \times \frac{G_0^q}{\triangle^q G_0},$$

where G_0 is compact connected simple Lie group and $\triangle^m G_0$ is the diagonal subgroup in G_0^m, for $m = p, q, n$. The spaces $\frac{G_0^n}{\triangle^n G_0}$ are n-symmetric spaces and it has been proved by Kowalski that under some conditions they are not k-symmetric for $k < n$ (see e.g. [27]). Hence, throughout this Chapter we shall designate such a space by a Kowaslki n-symmetric space. McKenzie Y. Wang and Wolfgang Ziller have shown that these spaces are standard Einstein manifolds ([45], [36]). We obtain new Einstein metrics with totally geodesic fibers. In section 1 we describe the isotropy subspaces and compute the necessary eigenvalues to obtain the Ricci curvature of an adapted metric on M. In Section 2 we show that, for $n > 4$, there exists at least one non-standard Einstein adapted metric on M, which is binormal or such that the metric on the base space is also Einstein if and only if $p = q$. We prove that for $n = 4$ the standard metric is the only Einstein adapted metric. We remark that for $n = 4$ the fibration above is a bisymmetric fibration of non-maximal rank and whose base space is isotropy reducible in opposition to the cases studied in Chapter 3.

4.1 Kowalski N-Symmetric Spaces - The Isotropy Representation and the Casimir Operators

Let G_0 be a compact connected simple Lie group and $\mathfrak{g}_0$ its Lie algebra. For any positive integer m we denote by G_0^m (or $\mathfrak{g}_0^m$) the direct product of G_0 ($\mathfrak{g}_0$, resp.) by itself m times. By $\triangle^m G_0$ (or $\triangle^m \mathfrak{g}_0$) we denote the diagonal in G_0^m (in $\mathfrak{g}_0^m$, resp.). Clearly, the Lie algebras of G_0^m and $\triangle^m G_0$ are $\mathfrak{g}_0^m$ and $\triangle^m \mathfrak{g}_0$, respectively.

Let n, p, q be positive integers such that $p + q = n$ and $2 \leq p \leq q \leq n - 2$. Set $G = G_0^n$ and consider the following closed subgroups of G:

$$K = \triangle^p G_0 \times \triangle^q G_0,$$

$$L = \triangle^n G_o \subset K.$$

The Lie algebras of G, K and L are, respectively,

$$\mathfrak{g} = \mathfrak{g}_0^n,$$

$$\mathfrak{k} = \triangle^p \mathfrak{g}_0 \times \triangle^q \mathfrak{g}_0,$$

$$\mathfrak{l} = \triangle^n \mathfrak{g}_0.$$

Following the notation of previous chapters, we write $M = G/L$, $N = G/K$ and $F = K/L$. We consider the fibration $F \to M \to N$. We note that $N = \frac{G_0^p}{\triangle^p G_0} \times \frac{G_0^q}{\triangle^q G_0}$ and thus this fibration is

$$\frac{G_0^n}{\triangle^n G_0} \to \frac{G_0^p}{\triangle^p G_0} \times \frac{G_0^q}{\triangle^q G_0}$$

with fiber

$$F = \frac{\triangle^p \mathfrak{g}_0 \times \triangle^q G \mathfrak{g}_0}{\triangle^n G_0}.$$

Let Φ_0 be the Killing form of $\mathfrak{g}_0$. Then, the Killing form of $\mathfrak{g}$ is

$$\Phi = \underbrace{\Phi_0 + \ldots + \Phi_0}_{n}.$$

Following the notation of chapters 2 and 3, let $\mathfrak{n}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ and $\mathfrak{p}$ be an orthogonal complement of $\mathfrak{l}$ in $\mathfrak{k}$, with respect to Φ. Then,

$$\mathfrak{g} = \underbrace{\mathfrak{l} \oplus \mathfrak{p}}_{\mathfrak{k}} \oplus \mathfrak{n}$$

and we write $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{n}$.

Lemma 4.1. *(i)* $\mathfrak{p} = \{(\underbrace{qX, \ldots, qX}_{p}, \underbrace{-pX, \ldots - pX}_{q}) : X \in \mathfrak{g}_0\}$ *and* $\mathfrak{p}$ *is an irreducible* $Ad\, L$*-module;*

(ii) $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$*, where*

$$\mathfrak{n}_1 = \{(X_1, \ldots, X_p, 0, \ldots, 0) : X_j \in \mathfrak{g}_0, \sum_{j=1}^{p} X_j = 0\} \subset \mathfrak{g}_0^p \times 0_q$$

$$\mathfrak{n}_2 = \{(0, \ldots, 0, X_1, \ldots, X_q) : X_j \in \mathfrak{g}_0, \sum_{j=1}^{q} X_j = 0\} \subset 0_p \times \mathfrak{g}_0^q;$$

Proof: Let $(\underbrace{Z,\ldots,Z}_{n}) \in \mathfrak{l}$ and $(\underbrace{X,\ldots,X}_{p},\underbrace{Y,\ldots,Y}_{q}) \in \mathfrak{k}$, where X, Y and Z are arbitrary elements in $\mathfrak{g}_0$.

$$\begin{aligned} 0 = & \ \Phi((X,\ldots,X,Y,\ldots,Y),(Z,\ldots,Z)) \\ = & \ p\Phi_0(X,Z) + q\Phi_0(Y,Z) \\ = & \ \Phi_0(pX+qY,Z) \end{aligned}$$

Since Φ_0 is non-degenerate on $\mathfrak{g}_0$ and Z is arbitrary, the identity above is possible if and only if $pX + qY = 0$. Hence, we conclude that $\mathfrak{p}$ is formed by elements of the form

$$(\underbrace{qX,\ldots,qX}_{p},\underbrace{-pX,\ldots-pX}_{q}),$$

where $X \in \mathfrak{g}_0$. Moreover, it is clear that $Ad\,L$-submodules of $\mathfrak{p}$ correspond to ideals of $\mathfrak{g}_0$. Since $\mathfrak{g}_0$ is simple we conclude that $\mathfrak{p}$ is irreducible.

Clearly, we may write $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, where $\mathfrak{n}_1$ is an orthogonal complement of $\triangle^p\mathfrak{g}_0 \times 0$ in $\mathfrak{g}_0^p \times 0$ and $\mathfrak{n}_2$ is an orthogonal complement of $0 \times \triangle^q\mathfrak{g}_0$ in $0 \times \mathfrak{g}_0^q$, with respect to Φ.
Let $(X_1,\ldots,X_p,0,\ldots,0) \in \mathfrak{g}_0^p \times 0$ and $(Z,\ldots,Z,0,\ldots,0) \in \triangle^p\mathfrak{g}_0 \times 0$, where Z and X_j are arbitrary elements in $\mathfrak{g}_0$.

$$\begin{aligned} 0 = & \ \Phi((X_1,\ldots,X_p,0,\ldots,0),(Z,\ldots,Z,0,\ldots,0)) \\ = & \ \Phi_0(X_1,Z) + \ldots + \Phi_0(X_p,Z) \\ = & \ \Phi_0(\textstyle\sum_{j=1}^p X_j, Z). \end{aligned}$$

Also by the nondegeneracy of Φ_0, we conclude that the identity above holds if and only if $\sum_{j=1}^p X_j = 0$. This gives the required expression for $\mathfrak{n}_1$. To prove the expression for $\mathfrak{n}_2$ is similar.

□

As usual we denote the Casimir operator of a subspace V with respect to Φ, the Killing form of $\mathfrak{g}$, by C_V (see Definition 1.3). Also, we denote the identity map and the zero map of V, by Id_V and 0_V, respectively.

Proposition 4.1. *(i)* $C_{\mathfrak{g}} = Id_{\mathfrak{g}}$;

(ii) $C_{\mathfrak{l}} = \dfrac{1}{n} Id_{\mathfrak{g}}$;

(iii) $C_{\mathfrak{p}} = \dfrac{q}{np} Id_{\mathfrak{g}_0^p} \times \dfrac{p}{nq} Id_{\mathfrak{g}_0^q}$;

(iv) $C_{\mathfrak{k}} = \dfrac{1}{p} Id_{\mathfrak{g}_0^p} \times \dfrac{1}{q} Id_{\mathfrak{g}_0^q}$;

(v) $C_{\mathfrak{n}_1} = \left(1 - \dfrac{1}{p}\right) Id_{\mathfrak{g}_0^p} \times 0_{\mathfrak{g}_0^q}$ *and* $C_{\mathfrak{n}_2} = 0_{\mathfrak{g}_0^p} \times \left(1 - \dfrac{1}{q}\right) Id_{\mathfrak{g}_0^q}$.

Proof: Let $(u_i)_i$ and $(u_i')_i$ be bases of $\mathfrak{g}_0$ dual with respect to Φ_0. Then the Casimir operator of $\mathfrak{g}_0$ with respect to Φ_0 is $C_{\mathfrak{g}_0} = \sum_i ad_{u_i} ad_{u_i'}$. We observe that since $\mathfrak{g}_0$ is a simple Lie algebra, $C_{\mathfrak{g}_0} = Id_{\mathfrak{g}_0}$.

(i)

$$\{(u_i, 0, \ldots, 0), (0, u_i, \ldots, 0), \ldots, (0, 0, \ldots, u_i)\}_i$$

and

$$\{(u_i', 0, \ldots, 0), (0, u_i', \ldots, 0), \ldots, (0, 0, \ldots, u_i')\}_i$$

are bases for $\mathfrak{g}$. We have

$$\Phi((0, \ldots, \underbrace{u_i}_{k^{th}}, \ldots, 0), (0, \ldots, \underbrace{u_j'}_{l^{th}}, \ldots, 0)) = \delta_{kl}\Phi_0(u_i, u_j') = \delta_{kl}\delta_{ij}.$$

Hence, the bases above are dual for Φ and thus we may write

$$\begin{aligned} C_{\mathfrak{g}} = &\ \textstyle\sum_i ad_{(u_i,0,\ldots,0)} ad_{(u_i',0,\ldots,0)} + \ldots + \sum_i ad_{(0,\ldots,0,u_i)} ad_{(0,\ldots,0,u_i')} \\ = &\ (\textstyle\sum_i ad_{u_i} ad_{u_i'}, 0, \ldots, 0) + \ldots + (0, 0, \ldots, \sum_i ad_{u_i} ad_{u_i'}) \\ = &\ (C_{\mathfrak{g}_0}, \ldots, C_{\mathfrak{g}_0}) \\ = &\ (Id_{\mathfrak{g}_0}, \ldots, Id_{\mathfrak{g}_0}) \\ = &\ Id_{\mathfrak{g}}. \end{aligned}$$

(ii) $\{(\underbrace{u_i, \ldots, u_i}_{n})\}_i$ and $\{(\underbrace{u_i', \ldots, u_i'}_{n})\}_i$ are bases for $\mathfrak{l}$ and we have

$$\Phi((u_i, \ldots, u_i), (u_j', \ldots, u_j')) = n\Phi_0(u_i, u_j') = n\delta_{ij}.$$

Hence, $\{\frac{1}{\sqrt{n}}(\underbrace{u_i, \ldots, u_i}_{n})\}_i$ and $\{\frac{1}{\sqrt{n}}(\underbrace{u_i', \ldots, u_i'}_{n})\}_i$ are bases for $\mathfrak{l}$ dual with respect to Φ. So we have

$$\begin{aligned} C_{\mathfrak{l}} = & \ \frac{1}{n}\textstyle\sum_i ad_{(u_i,\ldots,u_i)}ad_{(u'_i,\ldots,u'_i)} \\ = & \ \frac{1}{n}(\textstyle\sum_i ad_{u_i}ad_{u'_i},\ldots,\sum_i ad_{u_i}ad_{u'_i}) \\ = & \ \frac{1}{n}(C_{\mathfrak{g}_0},\ldots,C_{\mathfrak{g}_0}) \\ = & \ \frac{1}{n}(Id_{\mathfrak{g}_0},\ldots,Id_{\mathfrak{g}_0}) \\ = & \ \frac{1}{n}Id_{\mathfrak{g}}. \end{aligned}$$

(iii) $\{(\underbrace{qu_i,\ldots,qu_i}_{p},\underbrace{-pu_i,\ldots,-pu_i}_{q})\}_i$ and $\{(\underbrace{qu'_i,\ldots,qu'_i}_{p},\underbrace{-pu'_i,\ldots,-pu'_i}_{q})\}_i$ are bases of $\mathfrak{p}$.

$$\begin{aligned} & \Phi((qu_i,\ldots,qu_i,-pu_i,\ldots,-pu_i),(qu'_j,\ldots,qu'_j,-pu'_j,\ldots,-pu'_j)) \\ = & \ q^2p\Phi_0(u_i,u'_j)+p^2q\Phi_0(u_i,u'_j) \\ = & \ (p+q)pq\delta_{ij} \\ = & \ npq\delta_{ij}. \end{aligned}$$

Hence dual bases of $\mathfrak{p}$ for Φ are as follows:

$$\left\{\left(\left(\tfrac{q}{np}\right)^{\frac{1}{2}}u_i,\ldots,\left(\tfrac{q}{np}\right)^{\frac{1}{2}}u_i,-\left(\tfrac{p}{nq}\right)^{\frac{1}{2}}u_i,\ldots,-\left(\tfrac{p}{nq}\right)^{\frac{1}{2}}u_i\right)\right\}_i$$

$$\text{and }\left\{\left(\left(\tfrac{q}{np}\right)^{\frac{1}{2}}u'_i,\ldots,\left(\tfrac{q}{np}\right)^{\frac{1}{2}}u'_i,-\left(\tfrac{p}{nq}\right)^{\frac{1}{2}}u'_i,\ldots,-\left(\tfrac{p}{nq}\right)^{\frac{1}{2}}u'_i\right)\right\}_i.$$

Similar calculations as those done above show that

$$C_{\mathfrak{p}} = \frac{q}{np}(C_{\mathfrak{g}_0},\ldots,C_{\mathfrak{g}_0})\times\frac{p}{nq}(C_{\mathfrak{g}_0},\ldots,C_{\mathfrak{g}_0}) = \frac{q}{np}Id_{\mathfrak{g}_0^p}\times\frac{p}{nq}Id_{\mathfrak{g}_0^q}.$$

(iv)

$$C_{\mathfrak{k}} = C_{\mathfrak{l}}+C_{\mathfrak{p}} = \left(\frac{1}{n}+\frac{q}{np}\right)Id_{\mathfrak{g}_0^p}\times\left(\frac{1}{n}+\frac{p}{nq}\right)Id_{\mathfrak{g}_0^q} = \frac{1}{p}Id_{\mathfrak{g}_0^p}\times\frac{1}{q}Id_{\mathfrak{g}_0^q}.$$

(v)

$$C_{\mathfrak{n}_1}+C_{\mathfrak{n}_2} = C_{\mathfrak{n}} = C_{\mathfrak{g}}-C_{\mathfrak{k}} = \left(1-\frac{1}{p}\right)Id_{\mathfrak{g}_0^p}\times\left(1-\frac{1}{q}\right)Id_{\mathfrak{g}_0^q}$$

Since $[\mathfrak{n}_1,\mathfrak{g}]\subset\mathfrak{g}_0^p\times 0$ and $[\mathfrak{n}_2,\mathfrak{g}]\subset 0\times\mathfrak{g}_0^q$, we conclude that

$$C_{\mathfrak{n}_1} = \left(1 - \frac{1}{p}\right) Id_{\mathfrak{g}_0^p} \times 0_{\mathfrak{g}_0^q} \text{ and } C_{\mathfrak{n}_2} = 0_{\mathfrak{g}_0^p} \times \left(1 - \frac{1}{q}\right) Id_{\mathfrak{g}_0^q}.$$

□

For the eigenvalues of the Casimir operators of $\mathfrak{l}$, $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{n}$ we use notation similar to that used in previous chapters. We recall that $c_{\mathfrak{l},\mathfrak{p}}$ is the eigenvalue of $C_{\mathfrak{l}}$ on $\mathfrak{p}$, $c_{\mathfrak{k},i}$ is the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{n}_i$. The Casimir operator of $\mathfrak{p}$ is scalar on $\mathfrak{n}_i$, as we can see from Corollary 4.1, and b^i denotes the eigenvalue of $C_{\mathfrak{p}}$ on $\mathfrak{n}_i$, for $i = 1, 2$. Also $c_{\mathfrak{n}_i,\mathfrak{p}}$ and γ are the constants defined by

$$\Phi(C_{\mathfrak{n}_i}\cdot, .) \mid_{\mathfrak{p}\times\mathfrak{p}} = c_{\mathfrak{n}_i,\mathfrak{p}}\Phi \mid_{\mathfrak{p}\times\mathfrak{p}},\ i = 1, 2 \tag{4.1}$$

$$\Phi(C_{\mathfrak{k}}\cdot, .) \mid_{\mathfrak{p}\times\mathfrak{p}} = \gamma\Phi \mid_{\mathfrak{p}\times\mathfrak{p}} . \tag{4.2}$$

Corollary 4.1. *(i)* $c_{\mathfrak{l},\mathfrak{p}} = \dfrac{1}{n}$*;*
(ii) $C_{\mathfrak{p}}$ *is scalar on* $\mathfrak{n}_j$*,* $j = 1, 2$ *and* $b^1 = \dfrac{q}{np}$ *and* $b^2 = \dfrac{p}{nq}$*;*
(iii) $c_{\mathfrak{k},1} = \dfrac{1}{p}$ *and* $c_{\mathfrak{k},2} = \dfrac{1}{q}$*;*
(iv) $\gamma = \dfrac{q^2 + p^2}{npq}$*;*
(v) $c_{\mathfrak{n}_1,\mathfrak{p}} = \dfrac{(p-1)q}{pn}$ *and* $c_{\mathfrak{n}_2,\mathfrak{p}} = \dfrac{(q-1)p}{qn}$.

Proof: The number $c_{\mathfrak{l},\mathfrak{p}}$ is the eigenvalue of the Casimir operator of $\mathfrak{l}$ on $\mathfrak{p}$. Thus, it follows from Proposition 4.1 (ii), that $c_{\mathfrak{l},\mathfrak{p}} = \dfrac{1}{n}$. Since $\mathfrak{n}_1 \subset \mathfrak{g}_0^p \times 0$ and $\mathfrak{n}_2 \subset 0 \times \mathfrak{g}_0^q$, we conclude from Proposition 4.1 (iii) that $C_{\mathfrak{p}}$ is scalar on $\mathfrak{n}_1$ and on $\mathfrak{n}_2$; moreover its eigenvalues on these two spaces are $b^1 = \dfrac{q}{np}$ and $b^2 = \dfrac{p}{nq}$, respectively. Similarly, it follows from Proposition 4.1 (iv) that the eigenvalues of $C_{\mathfrak{k}}$ on $\mathfrak{n}_1$ and on $\mathfrak{n}_2$ are $c_{\mathfrak{k},1} = \dfrac{1}{p}$ and $c_{\mathfrak{k},2} = \dfrac{1}{q}$, respectively. Now to show (iv) we recall that γ is defined by the identity

$$\Phi(C_{\mathfrak{k}}\cdot, \cdot) \mid_{\mathfrak{p}\times\mathfrak{p}} = \gamma\Phi \mid_{\mathfrak{p}\times\mathfrak{p}} .$$

Let $(\underbrace{qX, \ldots, qX}_{p}, \underbrace{-pX, \ldots - pX}_{q}) \in \mathfrak{p}$.

$$\begin{aligned} & \Phi((qX, \ldots, qX, -pX, \ldots - pX), (qX, \ldots, qX, -pX, \ldots - pX)) \\ = \ & (q^2.p + p^2.q)\, \Phi_0(X, X) \\ = \ & npq\Phi_0(X, X) \end{aligned}$$

By using Proposition 4.1 (iv) we get the following:

$$
\begin{aligned}
& \Phi(C_{\mathfrak{k}}(qX,\dots,qX,-pX,\dots-pX),(qX,\dots,qX,-pX,\dots-pX)) \\
= \;& \Phi((\tfrac{q}{p}X,\dots,\tfrac{q}{p}X,-\tfrac{p}{q}X,\dots-\tfrac{p}{q}X),(qX,\dots,qX,-pX,\dots-pX)) \\
= \;& \left(\tfrac{q^2}{p}.p+\tfrac{p^2}{q}.q\right)\Phi_0(X,X) \\
= \;& (p^2+q^2)\Phi_0(X,X)
\end{aligned}
$$

Therefore, $\gamma = \dfrac{q^2+p^2}{npq}$.

Finally, for $j=1,2$, the numbers $c_{\mathfrak{n}_j,\mathfrak{p}}$ are defined by

$$\Phi(C_{\mathfrak{n}_j}\cdot,\cdot)\mid_{\mathfrak{p}\times\mathfrak{p}} = c_{\mathfrak{n}_j,\mathfrak{p}}\Phi\mid_{\mathfrak{p}\times\mathfrak{p}}.$$

From Proposition 4.1 (v), we obtain the following:

$$
\begin{aligned}
& \Phi(C_{\mathfrak{n}_1}(qX,\dots,qX,-pX,\dots-pX),(qX,\dots,qX,-pX,\dots-pX)) \\
= \;& \Phi\left(\left((1-\tfrac{1}{p})qX,\dots,(1-\tfrac{1}{p})qX,0,\dots,0\right),(qX,\dots,qX,-pX,\dots-pX)\right) \\
= \;& \frac{(p-1)q^2}{p}.p\Phi_0(X,X) \\
= \;& (p-1)q^2\Phi_0(X,X)
\end{aligned}
$$

Therefore, $c_{\mathfrak{n}_1,\mathfrak{p}} = \dfrac{(p-1)q^2}{npq} = \dfrac{(p-1)q}{np}$. Similarly, we show that $c_{\mathfrak{n}_2,\mathfrak{p}} = \dfrac{(q-1)p}{qn}$.
□

4.2 Existence of Einstein Adapted Metrics

In this Section we investigate the existence of adapted metrics on M with respect to the fibration

$$M = \frac{G_0^n}{\triangle^n G_0} \to \frac{G_0^p}{\triangle^p G_0} \times \frac{G_0^q}{\triangle^q G_0},$$

as in previous Section. We shall consider adapted metrics of the form

$$g_M = g_M(\lambda,\mu_1,\mu_2) \tag{4.3}$$

with respect to the decomposition $\mathfrak{m} = \mathfrak{p}\oplus\mathfrak{n}_1\oplus\mathfrak{n}_2$, i.e., g_M is induced by the scalar product

$$\lambda B\mid_{\mathfrak{p}\times\mathfrak{p}} \oplus \mu_1 B\mid_{\mathfrak{n}_1\times\mathfrak{n}_1} \oplus \mu_2 B\mid_{\mathfrak{n}_2\times\mathfrak{n}_2} \tag{4.4}$$

on $\mathfrak{m}$, where $B = -\Phi$, the negative of the Killing form. We observe that $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are inequivalent $Ad\,K$-modules, but they are not irreducible, for $n > 4$. Hence, adapted metrics on M are not necessarily of the form (4.3). However, throughout we shall focus only on adapted metrics of the form $g_M = g_M(\lambda, \mu_1, \mu_2)$, unless it is explicitly stated otherwise.

We recall that it has been proved by McKenzie Y. Wang and Wolfgang Ziller ([45], [36]) that the homogeneous space $M = \frac{G_0^n}{\triangle^n G_0}$, $n \geq 2$, is a standard Einstein manifold. Hence, there exists at least one Einstein adapted metric of the form (4.3). Indeed, by Corollary 1.2 and Corollary 4.1, the Ricci curvature of the standard metric is simply

$$Ric = \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{l},\mathfrak{m}}\right) B = \left(\frac{1}{4} + \frac{1}{2n}\right) B. \tag{4.5}$$

Consequently, the standard metric is Einstein. The standard metric is an example of a binormal metric on M with respect to the fibration $M \to N$. Below we shall classify all the binormal Einstein metrics on M.

Although the submodules $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are not $Ad\,K$-irreducible for $n > 4$, the Casimir operators of $\mathfrak{l}$, $\mathfrak{k}$ and $\mathfrak{p}$ are always scalar on $\mathfrak{n}_1$ and on $\mathfrak{n}_2$. Hence, it is enough to consider one irreducible submodule in $\mathfrak{n}_1$ and one irreducible submodule in $\mathfrak{n}_2$, for effects of Ricci curvature. Therefore, according to Theorem 2.1 (2.19), there is an one-to-one correspondence, up to homothety, between binormal adapted Einstein metrics on M and positive solutions of the following set of equations:

$$\delta_{12}^{\mathfrak{k}}(1 - X) = \delta_{12}^{\mathfrak{l}} \tag{4.6}$$

$$(\gamma + 2c_{\mathfrak{l},\mathfrak{p}})X^2 - (1 + 2c_{\mathfrak{k},j})\, X + (1 - \gamma + 2b^j) = 0,\ j = 1, 2 \tag{4.7}$$

Given a positive solution X, then binormal adapted Einstein metrics are given, up to homothety, by

$$g_M = g_M(1, X),$$

i.e., are induced by scalar products of the form $<,>= B\mid_{\mathfrak{p}\times\mathfrak{p}} \oplus XB\mid_{\mathfrak{n}\times\mathfrak{n}}$ on $\mathfrak{m}$.

Theorem 4.1. *Let us consider the fibration*

$$M = \frac{G_0^n}{\triangle^n G_0} \to \frac{G_0^p}{\triangle^p G_0} \times \frac{G_0^q}{\triangle^q G_0} = N,$$

where $p + q = n$ *and* $2 \leq p \leq q \leq n - 2$.

If $p \neq q$ or $n = 4$, then there exists on M precisely one binormal Einstein metric, up to homothety, which is the standard metric. For $n > 4$ and $p = q$, then there are on M precisely two binormal Einstein metrics, up to homothety, which are the standard metric and the metric induced by the scalar product

$$B\mid_{\mathfrak{p}\times\mathfrak{p}} \oplus \frac{n}{4} B\mid_{\mathfrak{n}\times\mathfrak{n}} .$$

Proof: From Corollary 4.1 we obtain that $\delta^{\mathfrak{k}}_{12} = c_{\mathfrak{k}_1} - c_{\mathfrak{k},2} = \frac{1}{p} - \frac{1}{q}$ whereas $\delta^{\mathfrak{l}}_{12} = c_{\mathfrak{l}_1} - c_{\mathfrak{l},2} = \frac{1}{n} - \frac{1}{n} = 0$. Hence, Equation (4.6) implies that $X = 1$ or $p = q$. So if $p \neq q$, if there exists a binormal Einstein metric it must be the standard metric. This we already know it is Einstein by [36]. Therefore, if $p \neq q$, then there exists, up to homothety, exactly one binormal Einstein metric on G/L which is the standard one.

By using Corollary 4.1 Equation (4.7) may be rewritten as

$$nX^2 - q(p+2)X + pq + q - p = 0, \text{ for } j = 1 \tag{4.8}$$

and

$$nX^2 - p(q+2)X + pq + p - q = 0, \text{ for } j = 2 \tag{4.9}$$

It is clear that $X = 1$ is actually a solution of both equations, and thus the standard metric is in fact Einstein.

Now suppose that $p = q$. As $n = p + q$, then $p = q = \frac{n}{2}$. Therefore, (4.8) and (4.9) become equivalent to

$$4X^2 - (n+4)X + n = 0 \tag{4.10}$$

The polynomial above has two positive roots, 1 and $\frac{n}{4}$. Therefore, for $p = q$ and $n > 4$, there exist precisely two binormal Einstein metrics.
□

Theorem 4.2. *Let g_M be an Einstein adapted metric on M of the form $g_M(\lambda; \mu_1, \mu_2)$. The projection g_N onto the base space is also Einstein if and only if $p = q$ and g_M is binormal.*

Proof: By Theorem 2.2 we know that if g_M and g_N are Einstein then we must have the relation

$$\frac{r_1}{r_2} = \left(\frac{b^1}{b^2}\right)^{\frac{1}{2}}. \tag{4.11}$$

From Lemma 4.1 (ii) we obtain $\left(\frac{b^1}{b^2}\right)^{\frac{1}{2}} = \frac{q}{p}$. Since $[\mathfrak{n}_1, \mathfrak{n}_2] = 0$, from Corollary 2.2, we get $r_i = \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{k},i}\right)$, $i = 1, 2$. Hence, $\frac{r_1}{r_2} = \frac{(p+2)q}{(q+2)p}$, by using Lemma 4.1 (iii). Therefore, (4.11) is possible if and only if $p = q$. Also from the proof of Theorem 2.2, if g_N and g_M are Einstein, then $\frac{\mu_1}{\mu_2} = \left(\frac{b^1}{b^2}\right)^{\frac{1}{2}} = \frac{p}{q} = 1$ and g_M is binormal. Conversely, if g_M is binormal and $p = q = n/2$, then by the above we also get

$$\frac{r_1}{\mu_1} = \frac{r_2}{\mu_2}$$

and g_N is Einstein.

□

We observe that since $\mathfrak{p}$ is an irreducible $Ad\,L$-submodule, g_F is always Einstein. Theorems 4.1 and 4.2 classify all Einstein binormal metrics on M and all Einstein adapted metrics such that g_N is also Einstein. It still remains to understand if there are other Einstein adapted metrics besides these. The Einstein equations in general for arbitrary p and q are extremely complicated. However with the help of Maple it is still possible to solve the problem in general. Next we shall classify all the Einstein adapted metrics on M of the form $g_M(\lambda, \mu_1, \mu_2)$.

Lemma 4.2. *Consider the fibration*

$$M = \frac{G_0^n}{\triangle^n G_0} \to \frac{G_0^p}{\triangle^p G_0} \times \frac{G_0^q}{\triangle^q G_0} = N,$$

where $p + q = n$ and $2 \leq p \leq q \leq n - 2$. There is a one-to-one correspondence between Einstein adapted metrics on M of the form $g_M = g_M(\lambda; \mu_1, \mu_2)$, up to homothety, and positive solutions of the following system of Equations:

$$-2q^2X_1^2 + nq(p+2)X_1 + 2p^2X_2^2 - np(q+2)X_2 = 0 \tag{4.12}$$
$$n^2 + q^2(p+1)X_1^2 + p^2(q-1)X_2^2 - nq(p+2)X_1 = 0. \tag{4.13}$$

To a positive solution (X_1, X_2) corresponds an Einstein metric of the form $g_M = g_M\left(1; \frac{1}{X_1}, \frac{1}{X_2}\right)$.

Proof: Let g_M be an adapted metric on M of the form $g_M(\lambda; \mu_1, \mu_2)$. We set

$$X_i = \frac{\lambda}{\mu_i},\ i = 1, 2. \tag{4.14}$$

Since the fiber F is irreducible we may use Proposition 2.1 to obtain the Ricci curvature. For $X \in \mathfrak{p}$,

$$Ric(X, X) = \left(\frac{1}{2}\left(c_{\mathfrak{l},\mathfrak{p}} + \frac{\gamma}{2}\right) + \frac{\lambda^2}{4}\sum_{j=1}^{n}\frac{c_{\mathfrak{n}_j,\mathfrak{p}}}{\mu_j^2}\right) B(X, X).$$

Hence by using the eigenvalues in Corollary 4.1 and the unknowns X_1 and X_2 defined in (4.14), we obtain

$$\frac{1}{2}\left(c_{\mathfrak{l},\mathfrak{p}} + \frac{\gamma}{2}\right) = \frac{1}{2n} + \frac{q^2+p^2}{4npq} = \frac{n}{4pq}$$

and

$$Ric(X,X) = \left(\frac{n}{4pq} + \frac{(p-1)q}{4pn}X_1^2 + \frac{(q-1)p}{4qn}X_2^2\right) B(X,X). \tag{4.15}$$

For $Y \in \mathfrak{n}_k$, the ricci curvature is given by

$$-\frac{\lambda}{2\mu_k}B(C_{\mathfrak{p}}Y,Y) + r_k B(Y,Y).$$

The Casimir operator of $\mathfrak{p}$ is scalar on $\mathfrak{n}_i$ with eigenvalues $\frac{q}{np}$, for $i=1$, and $\frac{p}{nq}$, for $i=2$, as we can see from Corollary 4.1. Since $[\mathfrak{n}_1,\mathfrak{n}_2]=0$, we use Corollary 2.2 to obtain r_k:

$$r_k = \frac{1}{2}\left(\frac{1}{2} + c_{\mathfrak{k},k}\right).$$

Hence, from Corollary 4.1, we get

$$Ric(Y,Y) = \left(-\frac{q}{2np}X_1 + \frac{p+2}{4p}\right) B(Y,Y),\ Y \in \mathfrak{n}_1 \tag{4.16}$$

$$Ric(Y,Y) = \left(-\frac{p}{2nq}X_2 + \frac{q+2}{4q}\right) B(Y,Y),\ Y \in \mathfrak{n}_2. \tag{4.17}$$

Finally, as $C_{\mathfrak{n}_i}(\mathfrak{p}) \subset \mathfrak{k}$, for $i=1,2$, from Proposition 2.1, we conclude that $Ric(\mathfrak{p},\mathfrak{n})=0$. Therefore, the Einstein equations for g_M are just

$$\frac{n}{4pq} + \frac{(p-1)q}{4pn}X_1^2 + \frac{(q-1)p}{4qn}X_2^2 = \lambda E \tag{4.18}$$

$$-\frac{q}{2np}X_1 + \frac{p+2}{4p} = \mu_1 E \tag{4.19}$$

$$-\frac{p}{2nq}X_2 + \frac{q+2}{4q} = \mu_2 E \tag{4.20}$$

where E is the Einstein constant. We obtain the system of equations stated in this lemma by eliminating E from the system above and rearranging the resulting equations.

□

Theorem 4.3. *Let us consider the fibration*

$$M = \frac{G_0^n}{\triangle^n G_0} \to \frac{G_0^p}{\triangle^p G_0} \times \frac{G_0^q}{\triangle^q G_0} = N,$$

where $p+q=n$ and $2 \leq p \leq q \leq n-2$. If $n>4$, there exist on M exactly two Einstein adapted metrics of the form $g_M = g_M(\lambda, \mu_1, \mu_2)$ and one is the standard metric. For $n=4$ the only Einstein adapted metric is the standard one.
For $p=q$ the non-standard Einstein adapted metric is binormal.

Proof: According to Lemma 4.2, the Einstein equations for g_M are as follows:

$$-2q^2X_1^2 + nq(p+2)X_1 + 2p^2X_2^2 - np(q+2)X_2 = 0 \tag{4.21}$$
$$n^2 + q^2(p+1)X_1^2 + p^2(q-1)X_2^2 - nq(p+2)X_1 = 0. \tag{4.22}$$

By using Maple we obtain that the solutions of the system above are $X_1 = X_2 = 1$ and

$$X_1 = \alpha,\ X_2 = \left(\frac{-q^2(p+1)\alpha^2 + nq(p+2)\alpha - n^2}{p^2(q-1)}\right)^{\frac{1}{2}}, \tag{4.23}$$

where α is a root of the polynomial

$$t(Z) = 4q^2Z^3 - 4q(n+pq+2)Z^2 + n(q(q+2)(p+1)+n+8)Z - (q+3)n^2.$$

The solution $X_1 = X_2 = 1$ corresponds to a standard metric and we recover the result that M is an Einstein standard manifold. We are now interested in analysing the existence of other metrics. First we observe that from the expression for X_2 in (4.23) we conclude that,

$$X_2 \in \mathbb{R} \text{ if and only if } \alpha \in \left(\frac{n}{q(p+1)}, \frac{n}{q}\right).$$

For this we compute the roots of the polynomial $-q^2(p+1)\alpha^2 + nq(p+2)\alpha - n^2$, as in (4.23).
Simple calculations show that

$$t\left(\frac{n}{q}\right) = \frac{p(q-1)^2n^2}{q} > 0$$
$$t\left(\frac{n}{q(p+1)}\right) = -\frac{p(p+3)^2(q-1)n^2}{q(p+1)^3} < 0$$

and thus t has at least one (positive) root in the interval $\left(\frac{n}{q(p+1)}, \frac{n}{q}\right)$, by the Bolzano Theorem. By the explained above, to this root corresponds an Einstein adapted metric on M. Furthermore, we show that this root is unique and distinct from 1. From this we conclude that there exists a non-standard Einstein adapted metric on M. We observe that the derivative of t, $\frac{dt}{dZ}$ has no real zeros. Simple calculations show that the zeros of $\frac{dt}{dZ}$ are

$$\frac{n+pq+2}{3q} \pm \frac{\sqrt{\delta}}{6q},$$

where

$$\delta = (q+1)^2p^2 - (q-1)(3q^2+4q-8)p - (q-1)(3q^2+8q+16).$$

We show that $\delta < 0$. For $p = q$, $\delta = -2q^4 - 2q^3 + 8q^2 - 16q + 16 < 0$, for every $q \geq 2$. So we suppose that $p < q$. In this case, since $p^2 \leq (q-1)p$, we have

$$\begin{aligned} \delta \leq & \ (q-1)\big(-(2p+3)q^2 - (2p+8)q + (9p-16)\big) \\ < & \ (q-1)(-2p^3 - 5p^2 + p - 16) \\ < & \ 0, \end{aligned}$$

for every $p \geq 2$.
With this we conclude that $\frac{dt}{dZ}$ has no real zeros and thus the root of t found above is the unique real root of t. Moreover, we must guarantee that this root does not yield the solution $X_1 = X_2 = 1$. If $X_1 = X_2 = 1$, then $\alpha = 1$ is a root of t. This may be possible since $1 \in \left(\frac{n}{q(p+1)}, \frac{n}{q}\right)$. Since

$$t(1) = p(q+2)(q-1)(n-4),$$

and, consequently, $\alpha = 1$ is a root of t if and only if $n = 4$. By using (4.23) we get that if $X_1 = 1$ when $n = 4$, then $X_2 = 1$ as well. Since non-standard Einstein adapted metrics are given by pairs of the form (4.23), with $\alpha \neq 1$, we conclude that there exists a unique non-standard Einstein adapted metric of the form $g_M(\lambda, \mu_1, \mu_2)$ if and only if $n > 4$; in the case $n = 4$, the standard metric is the unique Einstein adapted metric of the form $g_M(\lambda, \mu_1, \mu_2)$. Finally, we observe that, if $n = 4$, the subspaces $\mathfrak{n}^1$ and $\mathfrak{n}^2$ are irreducible $Ad\, L$-submodules. Hence, any adapted metric on M is of the form $g_M(\lambda, \mu_1, \mu_2)$. Therefore, we conclude that, for $n = 4$, there exists a unique Einstein adapted metric on M and it is the standard one.
Since there is a unique non-standard Einstein adapted metric on M, it follows from Theorem 4.1 that this metric is binormal if and only if $p = q$.
$\square$

As it has been observed previously the modules $\mathfrak{n}_k$, $k = 1, 2$, are not irreducible $Ad\, K$-modules. From Lemma 4.1 we deduce a possible decomposition for $\mathfrak{n}_1$ and $\mathfrak{n}_2$ into irreducible $Ad\, K$-submodules:

Lemma 4.3. $\mathfrak{n}_1 = \oplus_{j=1}^{p-1}\mathfrak{n}_{1,j}$ *and* $\mathfrak{n}_2 = \oplus_{j=1}^{q-1}\mathfrak{n}_{2,j}$ *, where*

$$\mathfrak{n}_{1,j} = \{(\underbrace{X,\ldots,X}_{j}, -jX, 0, \ldots, 0) \in \mathfrak{g}_0^n : X \in \mathfrak{g}_0\},\ \textit{for } j = 1, \ldots, p-1$$

$$\mathfrak{n}_{2,j} = \{(\underbrace{0,\ldots,0}_{p}, \underbrace{X,\ldots,X}_{j}, -jX, 0, \ldots, 0) \in \mathfrak{g}_0^n : X \in \mathfrak{g}_0\},\ \textit{for } j = 1, \ldots, q-1.$$

Furthermore, the $\mathfrak{n}_{i,j}$*'s are irreducible* $Ad\,K$*-submodules.*

The fact that the modules above are irreducible follows from the fact that $\mathfrak{g}_0$ is simple. Also by similar calculations to those in Lemma 4.1 we obtain the Casimir operators of the submodules $\mathfrak{n}_{1,j}$ and $\mathfrak{n}_{2,j}$:

Lemma 4.4. $C_{\mathfrak{n}_{1,j}} = \Big(\underbrace{\frac{1}{j(j+1)}, \ldots, \frac{1}{j(j+1)}}_{j}, \frac{j}{j+1}, 0, \ldots, 0\Big)$*, for* $j = 1, \ldots, p-$ *1, and*
$C_{\mathfrak{n}_{2,j}} = \Big(\underbrace{0,\ldots,0}_{p}, \underbrace{\frac{1}{j(j+1)}, \ldots, \frac{1}{j(j+1)}}_{j}, \frac{j}{j+1}, 0, \ldots, 0\Big)$*, for* $j = 1, \ldots, q-1$.

Proof: $\{(\underbrace{u_i,\ldots,u_i}_{j}, -ju_i, 0\ldots,0\}_i$ and $\{(\underbrace{u_i',\ldots,u_i'}_{j}, -ju_i', 0, \ldots, 0)\}_i$ are bases of $\mathfrak{n}_{1,j}$.

$$\begin{aligned} & \Phi((u_i,\ldots,u_i,-ju_i,0,\ldots,0),(u_j',\ldots,u_j',-ju_j',0,\ldots,0)) \\ = \ & k\Phi_0(u_i,u_j') + k^2\Phi_0(u_i,u_j') \\ = \ & (k+1)k\delta_{ij} \end{aligned}$$

Hence dual bases of $\mathfrak{n}_{1,j}$ for Φ are as follows:
$\{\frac{1}{\sqrt{k(k+1)}}(u_i,\ldots,u_i,-ju_i,0\ldots,0\}_i$ and $\{\frac{1}{\sqrt{k(k+1)}}(u_i',\ldots,u_i',-ju_i',0,\ldots,0)\}_i$.
By using these bases we obtain the required expression for the Casimir operator of $\mathfrak{n}_{1,j}$. For $C_{\mathfrak{n}_{2,j}}$ is similar.
□
Hence, we may consider on M an adapted metric of the form

$$g_M = g_M(\lambda; \mu_{1,1}, \ldots, \mu_{1,p-1}, \mu_{2,1}, \ldots, \mu_{2,q-1}). \tag{4.24}$$

Clearly, whereas the submodules $\mathfrak{n}_{i,j}$ are $Ad\,K$-irreducible, the reader should note that they are not pairwise inequivalent. Hence, adapted metrics on M are not necessarily of the form (4.25).

Lemma 4.5. *Let g_M be an adapted metric on M of the form*

$$g_M(\lambda;\mu_{1,1},\ldots,\mu_{1,p-1},\mu_{2,1},\ldots,\mu_{2,q-1}).$$

If g_M is Einstein, then it is of the form

$$g_M(\lambda,\mu_1,\mu_2). \tag{4.25}$$

Proof: By Corollary 1.5, if exists on M an Einstein adapted metric, then

$$\Big(\sum_{j=1}^{p-1}\nu_{1,j}C_{\mathfrak{n}_{1,j}}+\sum_{j=1}^{q-1}\nu_{2,j}C_{\mathfrak{n}_{2,j}}\Big)(\mathfrak{p})\subset\mathfrak{k}, \tag{4.26}$$

for some $\nu_{1,j},\ \nu_{2,j}>0$. The inclusion (4.26) implies

$$\sum_j \nu_{i,j}C_{\mathfrak{n}_{i,j}}(\mathfrak{p})\subset\mathfrak{k}, \tag{4.27}$$

for $i=1,2$, since $C_{\mathfrak{n}_{1,j}}(\mathfrak{g})\subset\mathfrak{g}_0^p\times 0_q$ and $C_{\mathfrak{n}_{2,j}}(\mathfrak{g})\subset 0_p\times\mathfrak{g}_0^q$. For $k=1,\ldots,p$, let P_k denote the kth component of $P=\sum_j\nu_{1,j}C_{\mathfrak{n}_{1,j}}$. By 4.1 (vi), we have

$$P_k=\frac{k-1}{k}\nu_{1,k-1}+\frac{1}{k(k+1)}\sum_{j=k}^{p-1}\nu_{1,k},$$

for $k=2,\ldots,p-1$. We may then write

$$P_{k+1}=P_k+\frac{k-1}{k}(\nu_{1,k}-\nu_{1,k-1}). \tag{4.28}$$

On the other hand, the condition $P(\mathfrak{p})\subset\mathfrak{k}$, implies that $P_k=P_{k+1}$. Hence, from (4.28) we obtain that $\nu_{1,k}=\nu_{1,k-1}$. Similarly we show that $\nu_{2,k}=\nu_{2,k-1}$. Now from the proof of Corollary 1.5, we can see that for an adapted Einstein metric on M as in (4.3), then the constants $\nu_{1,k}=1/\mu_{1,k}^2$ and $\nu_{2,k}=1/\mu_{2,k}^2$ must satisfy (4.26). This concludes the proof.

□

Therefore, Theorem 4.2 may be understood as a classification of Einstein metrics of the form (4.25).

Corollary 4.2. *Let us consider the fibration*

$$M=\frac{G_0^n}{\triangle^n G_0}\to\frac{G_0^p}{\triangle^p G_0}\times\frac{G_0^q}{\triangle^q G_0}=N,$$

where $p+q=n$ and $2\le p\le q\le n-2$. Let

$$g_M(\lambda;\mu_{1,1},\ldots,\mu_{1,p-1},\mu_{2,1},\ldots,\mu_{2,q-1}) \tag{4.29}$$

be an adapted metric on M, corresponding to the decomposition of $\mathfrak{n}$ given in Lemma 4.3. If $n > 4$, there exist on M exactly two Einstein adapted metrics of the form (4.29) and one is the standard metric.

4.3 Closing Remarks

The main aim of this thesis was to establish conditions for existence of homogeneous Einstein metrics with totally geodesic fibers and to bring new existence and non-existence results for a significant class of homogeneous spaces. Necessary and sufficient conditions for existence of such metrics were obtained under some hypothesis and, for some classes of homogeneous fibrations, new Einstein metrics with totally geodesic fibers were found and in other cases all such metrics were classified. Nevertheless, some questions remain open for further research.

For irreducible bisymmetric fibrations of maximal rank all the Einstein adapted metrics were classified in the case when G is an exceptional Lie group. If G is a classical group, all the Einstein adapted metrics were classified for Type I, whereas for Type II we classified only those whose restriction to the fiber is still Einstein. For these bisymmetric fibrations which admit an Einstein adapted metric which satisfies this condition we can still classify all the other Einstein adapted metrics. However, in this classical case, it remains to obtain a general classification of Einstein adapted metrics.

Furthermore, the techniques and results for irreducible bisymmetric fibrations of maximal rank suggest that a similar research can be developed for homogeneous fibrations whose fiber and base space are p-symmetric spaces of higher order. For instance, the classification of compact simply-connected 3-symmetric spaces would allow us to consider fibrations whose fiber is a 3-symmetric space and the base is still an irreducible symmetric space.

For Kowalski n-symmetric spaces we have shown that there exists a non-standard Einstein adapted metric, if $n > 4$. For $n = 4$ we have proved that the standard metric is the only Einstein metric with totally geodesic fibers. It remains to classify all the Einstein adapted metrics, if $n > 4$.

APPENDIX A

A classification of bisymmetric triples of maximal rank was given in Chapter 3 and they are listed in Tables 3.4, 3.5, 3.6 and 3.7. In this Appendix we determine the isotropy representation for each bisymmetric triple and compute the eigenvalues γ_a's and b_a^ϕ of the Casimir operators $C_{\mathfrak{k}}$ and $C_{\mathfrak{p}_a}$ along the subspaces $\mathfrak{p}_a$ and $\mathfrak{n}$, respectively. We use the formulas presented in Chapter 3 in Propositions 3.1 and 3.2. We shall systematically use the roots systems and the dual Coxeter numbers. The roots systems used can be found in [16] and [34] and the dual Coxeter numbers are given in Table 3.1.

As introduced in Chapter 3, if $\mathfrak{n} = \oplus_j \mathfrak{n}^j$ is a decomposition of $\mathfrak{n}$ into irreducible $Ad\, L$-modules, we write

$$\mathcal{R}_{\mathfrak{n}^j} = \{\phi \in \mathcal{R} : E_\phi \in (\mathfrak{n}^j)^{\mathbb{C}}\} \text{ and } \mathfrak{n}^j = < X_\phi, Y_\phi : \phi \in \mathcal{R}^+_{\mathfrak{n}^j} > . \tag{A.1}$$

We recall that b_a^ϕ, for $\phi \in \mathcal{R}_{\mathfrak{n}^j}$, is the eigenvalue of $C_{\mathfrak{p}_a}$ on $\mathfrak{n}^j$:

$$C_{\mathfrak{p}_a} |_{\mathfrak{n}^j} = b_a^\phi Id_{\mathfrak{n}^j}.$$

If $\mathfrak{p}$ is $Ad\, L$-irreducible, we write simply b^ϕ for the eigenvalue of $C_{\mathfrak{p}}$ on $\mathfrak{n}^j$. These eigenvalues are given by the formula from Proposition 3.1:

$$b_a^\phi = \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+_{\mathfrak{p}_a}} d_{\alpha\phi} |\alpha|^2, \tag{A.2}$$

where

$$d_{\alpha\phi} = q_{\alpha\phi} - p_{\alpha\phi} - 2p_{\alpha\phi} q_{\alpha\phi} \tag{A.3}$$

and $\phi + n\alpha$, $p_{\alpha\phi} \leq n \leq q_{\alpha\phi}$ is the α-series containing ϕ.

Also if $\mathfrak{k}_a$ is a simple ideal of $\mathfrak{k}$, γ_a is the eigenvalue of $C_{\mathfrak{k}}$ on $\mathfrak{k}_a$:

$$C_{\mathfrak{k}} |_{\mathfrak{k}_a} = \gamma_a Id_{\mathfrak{k}_a}.$$

And finally to compute the γ_a's, when $\mathfrak{k}_a$ is a simple ideal of $\mathfrak{k}$, we use Proposition 3.2:

$$\gamma_a = \frac{h^*(\mathfrak{k}_a)}{\delta_a . h^*(\mathfrak{g})},\ a = 1, \ldots, s \tag{A.4}$$

. where $h^*(\mathfrak{k}_a)$ and $h^*(\mathfrak{g})$ are the dual Coxeter numbers of $\mathfrak{k}_a$ and $\mathfrak{g}$, respectively, and $\delta_a = |\alpha|^2/|\beta|^2$, for α a long root of $\mathfrak{g}$ and β a long root of $\mathfrak{k}_a$. If there is only one root length on $\mathfrak{g}$ or both $\mathfrak{g}$ and $\mathfrak{k}_a$ have two root lengths, $\delta_a = 1$. If $\delta_a \neq 1$ then δ_a it is equal to 2 or 3. If $\mathfrak{p}$ is $Ad\,L$-irreducible, then we write simply γ for the eigenvalue of $C_{\mathfrak{k}}$ on the corresponding simple ideal of $\mathfrak{k}$.
Since the computations of the eigenvalues γ_a and b_a^ϕ consist of a systematic use of formulas (A.2) and (A.4) some details of these computations are omitted. For each simple Lie algebra $\mathfrak{g}$ we present the set of roots of $\mathfrak{g}$ and the corresponding length of the roots. We consider each symmetric pair of maximal rank $(\mathfrak{g}, \mathfrak{k})$ and present the subsets of roots of each simple ideal $\mathfrak{k}_a$ of $\mathfrak{k}$, the corresponding value of γ_a and the subset of roots of the isotropy space $\mathfrak{n}$. We recall that $\mathcal{R}_{\mathfrak{n}} = \mathcal{R} - \mathcal{R}_{\mathfrak{k}}$. Finally, for each bisymmetric triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$, we indicate the subset of roots $\mathcal{R}_{\mathfrak{p}}$ for the symmetric complement $\mathfrak{p}$ of the symmetric pair $(\mathfrak{k}, \mathfrak{l})$ and the subset of roots of each irreducible $Ad\,L$-submodule $\mathfrak{n}^i$ of $\mathfrak{n}$. For bisymmetric triples of Type I, we present the essential information to compute the eigenvalues b^ϕ on each subspace $\mathfrak{n}^i$. Since $\mathfrak{n}^i$ is $Ad\,L$-irreducible, it suffices to choose any root ϕ in $\mathcal{R}_{\mathfrak{n}^i}$. Thus, we choose a root ϕ in each $\mathfrak{n}^i$ and indicate all the roots $\alpha \in \mathcal{R}_{\mathfrak{p}}^+$ such that the string $\phi + n\alpha$ is not singular, i.e., it contains other roots besides ϕ; only for these α's the coefficients $d_{\alpha\phi}$ in formula (A.2) are non-zero. We indicate the elements in the non-singular string $\phi + n\alpha$; only three cases occur: this string is formed either by
(i) ϕ, $\phi + \alpha$, in which case $p_{\alpha\phi} = 0$ and $q_{\alpha\phi} = 1$; thus, $d_{\alpha\phi} = 1$;
(ii) ϕ, $\phi - \alpha$, in which case $p_{\alpha\phi} = -1$ and $q_{\alpha\phi} = 0$; thus, $d_{\alpha\phi} = 1$;
(iii) ϕ, $\phi \pm \alpha$, in which case $p_{\alpha\phi} = -1$ and $q_{\alpha\phi} = 1$; thus, $d_{\alpha\phi} = 4$.
We indicate the length $|\alpha|^2$ of each root α listed previously. Once all this information is obtained, we apply (A.2) to compute b^ϕ.
The reader may notice throughout that, in some cases, the root α indicated is not a positive root. We observe that since $d_{\alpha\phi} = d_{-\alpha\phi}$, we may choose the α's in $\mathcal{R}_{\mathfrak{p}}$ independently of the sign, as long as only one of $\pm\alpha$ is chosen. This allows us to chose the necessary α's without any considerations about the order of the roots in the Lie algebra $\mathfrak{g}$.
For bisymmetric triples of Type II, we obtain the eigenvalues b_a^ϕ from the bisymmetric triples of Type I. For instance, b_1^ϕ and b_2^ϕ for $(\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2, \mathbb{R} \oplus \mathbb{R})$ are given by b^ϕ of $(\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2, \mathbb{R} \oplus \mathfrak{su}_2)$ and by b^ϕ of $(\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2, \mathfrak{su}_2 \oplus \mathbb{R})$, respectively.

In the cases of the classical Lie algebras, it is easy to understand whether the sum of two roots is a root, due to the simplicity of their root systems. However, in the exceptional cases, this may be rather complicated, mainly in the cases of the Lie algebras $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$. In these three cases, auxiliary lemmas are provided, where conditions under which the sum of two roots is a root are stated. These are the Lemmas A.1, A.2 and A.3.

A.1 A_{n-1}

In this Section we consider bisymmetric triples of the form $(\mathfrak{su}_n, \mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}, \mathfrak{l})$, $1 \leq p \leq n-1$. We set $\mathfrak{k}_1 = \mathfrak{su}_p$, $\mathfrak{k}_2 = \mathfrak{su}_{n-p}$ and $\mathfrak{k}_0 = \mathbb{R}$.
For a root system of type A_{n-1} for $\mathfrak{g}$ we take

$$\mathcal{R} = \{\pm(e_i - e_j) : i \leq i < j \leq n\}. \tag{A.5}$$

In $\mathfrak{g}$ there is only one root length and is

$$|\alpha|^2 = \frac{1}{n}. \tag{A.6}$$

Symmetric Pair A.1. $(\mathfrak{su}_n, \mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R})$, $p = 1, \ldots, n-1$.

$\mathfrak{k}_i$	$\mathcal{R}_{\mathfrak{k}_i}$	γ_i
$\mathfrak{su}_p$	$\{\pm(e_i - e_j) : 1 \leq i < j \leq p\}$	$\frac{p}{n}$, $p \geq 2$
$\mathfrak{su}_{n-p}$	$\{\pm(e_i - e_j) : p+1 \leq i < j \leq n\}$	$\frac{n-p}{n}$, $p \leq n-2$

$$\mathcal{R}_{\mathfrak{n}} = \{\pm(e_i - e_j) : 1 \leq i \leq p,\, p+1 \leq j \leq n\}$$

Bisymmetric Triple A.1. $(\mathfrak{su}_n, \mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}, \mathfrak{su}_l \oplus \mathfrak{su}_{p-l} \oplus \mathbb{R} \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R})$, $1 \leq p \leq n-1$, $1 \leq l \leq p-1$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i - e_j) : 1 \leq i \leq l,\, l+1 \leq j \leq p\}\\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2\\
&\mathcal{R}_{\mathfrak{n}^1} = \{\pm(e_i - e_j) : 1 \leq i \leq l,\, p+1 \leq j \leq n\}\\
&\mathcal{R}_{\mathfrak{n}^2} = \{\pm(e_i - e_j) : l+1 \leq i \leq p,\, p+1 \leq j \leq n\}
\end{aligned}$$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* α*'s*	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$e_1 - e_n$	$e_1 - e_j$, $l+1 \leq j \leq p$	ϕ, $\phi - \alpha$	1	$p-l$	$\frac{1}{n}$	$\frac{p-l}{2n}$
$\mathfrak{n}^2$	$e_p - e_n$	$e_j - e_p$, $1 \leq j \leq l$	ϕ, $\phi + \alpha$	1	l	$\frac{1}{n}$	$\frac{l}{2n}$

Bisymmetric Triple A.2. $(\mathfrak{su}_n, \mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}, \mathfrak{su}_p \oplus \mathfrak{su}_s \oplus \mathfrak{su}_{n-p-s} \oplus \mathbb{R} \oplus \mathbb{R})$, $1 \leq p \leq n-1$, $1 \leq s \leq n-p-1$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i - e_j) : p+1 \leq i \leq p+s,\, p+s+1 \leq j \leq n\}\\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2\\
&\mathcal{R}_{\mathfrak{n}^1} = \{\pm(e_i - e_j) : 1 \leq i \leq p,\, p+1 \leq j \leq p+s\}\\
&\mathcal{R}_{\mathfrak{n}^2} = \{\pm(e_i - e_j) : 1 \leq i \leq p,\, p+s+1 \leq j \leq n\}
\end{aligned}$$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* α*'s*	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$e_1 - e_{p+s}$	$e_{p+s} - e_i$, $p+s+1 \leq j \leq n$	ϕ, $\phi + \alpha$	1	$n-p-s$	$\frac{1}{n}$	$\frac{n-p-s}{2n}$
$\mathfrak{n}^2$	$e_p - e_n$	$e_i - e_n$, $p+1 \leq j \leq p+s$	ϕ, $\phi - \alpha$	1	s	$\frac{1}{n}$	$\frac{s}{2n}$

Bisymmetric Triple A.3. $(\mathfrak{su}_n, \mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}, \mathfrak{su}_l \oplus \mathfrak{su}_{p-l} \oplus \mathfrak{su}_s \oplus \mathfrak{su}_{n-p-s} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R})$, $1 \leq p \leq n-1$, $1 \leq l \leq p-1$, $1 \leq s \leq n-p-1$. *(Type II)*

$$\mathcal{R}_{\mathfrak{p}_1} = \{\pm(e_i - e_j) : 1 \leq i \leq l,\, l+1 \leq j \leq p\}$$
$$\mathcal{R}_{\mathfrak{p}_2} = \{\pm(e_i - e_j) : p+1 \leq i \leq p+s,\, p+s+1 \leq j \leq n\}$$
$$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \mathfrak{n}^3 \oplus \mathfrak{n}^4$$

$\mathcal{R}_{\mathfrak{n}^i}$	b_1^ϕ	b_2^ϕ
$\{\pm(e_i - e_j) : 1 \leq i \leq l,\, p+1 \leq j \leq p+s\}$	$\frac{p-l}{2n}$	$\frac{n-p-s}{2n}$
$\{\pm(e_i - e_j) : 1 \leq i \leq 1,\, p+s+1 \leq j \leq n\}$	$\frac{p-l}{2n}$	$\frac{s}{2n}$
$\{\pm(e_i - e_j) : l+1 \leq i \leq p,\, p+1 \leq j \leq p+s\}$	$\frac{l}{2n}$	$\frac{n-p-s}{2n}$
$\{\pm(e_i - e_j) : l+1 \leq i \leq p,\, p+s+1 \leq j \leq n\}$	$\frac{l}{2n}$	$\frac{s}{2n}$

A.2 B_n

In this Section we consider all the bisymmetric triples of the form $(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{l})$, for $0 \leq p \leq n-1$.
A root system for $\mathfrak{g}$ is

$$\mathcal{R} = \{\pm e_i : 1 \leq i \leq n;\ \pm e_i \pm e_j : 1 \leq i < j \leq n\} \tag{A.7}$$

and the length of a root is

$$|\alpha|^2 = \begin{cases} \frac{1}{2(2n-1)}, & \alpha = \pm e_i \\ \frac{1}{2n-1}, & \alpha = \pm e_i \pm e_j \end{cases}. \tag{A.8}$$

Symmetric Pair A.2. $(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)})$, $0 \leq p \leq n-1$.

$\mathfrak{k}_i$	$\mathcal{R}_{\mathfrak{k}_i}$	γ_i
$\mathfrak{so}_{2p+1}$	$\{\pm e_i : 1 \leq i \leq p;\ \pm e_i \pm e_j : 1 \leq i < j \leq p\}$	$\frac{2p-1}{2n-1}$, $p \geq 1$
$\mathfrak{so}_{2(n-p)}$	$\{\pm e_i \pm e_j : p+1 \leq i < j \leq n\}$	$\frac{2(n-p-1)}{2n-1}$, $p \leq n-2$

$$\mathcal{R}_{\mathfrak{n}} = \{\pm e_i \pm e_j : 1 \leq i \leq,\ p+1 \leq j \leq n\}$$

Bisymmetric Triple A.4. $(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2l+1} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2(n-p)})$, $0 \leq p \leq n-1$, $0 \leq l \leq p-1$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}} = \{\pm e_i : l+1 \leq i \leq p,\ \pm e_i \pm e_j : 1 \leq i \leq l,\ l+1 \leq j \leq p\} \\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \\
&\mathcal{R}_{\mathfrak{n}^1} = \{\pm e_i : p+1 \leq i \leq n;\ \pm e_i \pm e_j : 1 \leq i \leq l,\ p+1 \leq j \leq n\} \\
&\mathcal{R}_{\mathfrak{n}^2} = \{\pm e_i \pm e_j : l+1 \leq i \leq p,\ p+1 \leq j \leq n\}
\end{aligned}$$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' ss$	$\|\alpha\|^2$	b^ϕ
$\mathfrak{n}^1$	e_n	e_i, $l+1 \leq i \leq p$	ϕ, $\phi \pm \alpha$	4	$p-l$	$\frac{1}{2(2n-1)}$	$\frac{p-l}{2n-1}$
$\mathfrak{n}^2$	$e_p + e_n$	$e_i + e_p$, $1 \leq i \leq l$ e_p	ϕ, $\phi - \alpha$	1	$2l$ 1	$\frac{1}{2n-1}$ $\frac{1}{2(2n-1)}$	$\frac{4l+1}{4(2n-1)}$

Bisymmetric Triple A.5. $(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2s} \oplus \mathfrak{so}_{2(n-p-s)})$, $0 \leq p \leq n-1$, $1 \leq s \leq n-p-1$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}} = \{\pm e_i \pm e_j : p+1 \leq i \leq p+s,\ p+s+1 \leq j \leq n\} \\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \\
&\mathcal{R}_{\mathfrak{n}^1} = \{\pm e_i : p+1 \leq i \leq p+s;\ \pm e_i \pm e_j : 1 \leq i \leq p,\ p+1 \leq j \leq p+s\} \\
&\mathcal{R}_{\mathfrak{n}^2} = \{\pm e_i : p+s+1 \leq i \leq n;\ \pm e_i \pm e_j : 1 \leq i \leq p,\ p+s+1 \leq j \leq n\}
\end{aligned}$$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' s$	$\|\alpha\|^2$	b^ϕ
$\mathfrak{n}^1$	e_{p+1}	$e_{p+1} \pm e_i$, $p+s+1 \leq i \leq n$	ϕ, $\phi - \alpha$	1	$2(n-p-s)$	$\frac{1}{2n-1}$	$\frac{n-p-s}{2n-1}$
$\mathfrak{n}^2$	e_n	$\pm e_i + e_n$, $p+1 \leq i \leq p+s$	ϕ, $\phi + \alpha$	1	$2s$	$\frac{1}{2n-1}$	$\frac{s}{2n-1}$

Bisymmetric Triple A.6. $(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2p+1} \oplus \mathfrak{u}_{n-p})$, $0 \leq p \leq n-1$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i + e_j) : p+1 \leq i < j \leq n\}$
$\mathfrak{n}$ *irreducible* $Ad\,L$*-module*

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' s$	$\lvert\alpha\rvert^2$	b^{ϕ}
e_n	$e_i + e_n$, $p+1 \leq i \leq n-1$	ϕ, $\phi - \alpha$	1	$n-p-1$	$\frac{1}{2n-1}$	$\frac{n-p-1}{2(2n-1)}$

Bisymmetric Triple A.7. $(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2l+1} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{u}_{n-p})$, $0 \leq p \leq n-1$, $0 \leq l \leq p-1$. *(Type II)*

$R_{\mathfrak{p}_1} = \{\pm e_i : l+1 \leq i \leq p,\ \pm e_i \pm e_j : 1 \leq i \leq l,\ l+1 \leq j \leq p\}$
$\mathcal{R}_{\mathfrak{p}_2} = \{\pm(e_i + e_j) : p+1 \leq i < j \leq n\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2$

$\mathcal{R}_{\mathfrak{n}^i}$	b_1^{ϕ}	b_2^{ϕ}
$\{\pm e_i : p+1 \leq i \leq n;\ \pm e_i \pm e_j : 1 \leq i \leq l,\ p+1 \leq j \leq n\}$	$\frac{p-l}{2n-1}$	$\frac{n-p-1}{2(2n-1)}$
$\{\pm e_i \pm e_j : l+1 \leq i \leq p,\ p+1 \leq j \leq n\}$	$\frac{4l+1}{4(2n-1)}$	$\frac{n-p-1}{2(2n-1)}$

Bisymmetric Triple A.8. $(\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2l+1} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2s} \oplus \mathfrak{so}_{2(n-p-s)})$, $0 \leq p \leq n-1$, $0 \leq l \leq p-1$, $1 \leq s \leq n-p-1$. *(Type II)*

$\mathcal{R}_{\mathfrak{p}_1} = \{\pm e_i : l+1 \leq i \leq p,\ \pm e_i \pm e_j : 1 \leq i \leq l,\ l+1 \leq j \leq p\}$,
$\mathcal{R}_{\mathfrak{p}_2} = \{\pm e_i \pm e_j : p+1 \leq i \leq p+s,\ p+s+1 \leq j \leq n\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \mathfrak{n}^3 \oplus \mathfrak{n}^4$

$\mathcal{R}_{\mathfrak{n}^i}$	b_1^{ϕ}	b_2^{ϕ}
$\{\pm e_i : p+1 \leq i \leq p+s;\ \pm e_i \pm e_j : 1 \leq i \leq l,\ p+1 \leq j \leq p+s\}$	$\frac{p-l}{2n-1}$	$\frac{n-p-s}{2n-1}$
$\{\pm e_i : p+s+1 \leq i \leq n;\ \pm e_i \pm e_j : 1 \leq i \leq l,\ p+s+1 \leq j \leq n\}$	$\frac{p-l}{2n-1}$	$\frac{s}{2n-1}$
$\{\pm e_i \pm e_j : l+1 \leq i \leq p,\ p+1 \leq j \leq p+s\}$	$\frac{4l+1}{4(2n-1)}$	$\frac{n-p-s}{2n-1}$
$\{\pm e_i \pm e_j : l+1 \leq i \leq p,\ p+s+1 \leq j \leq n\}$	$\frac{4l+1}{4(2n-1)}$	$\frac{s}{2n-1}$

A.3 D_n

In this Section we consider all the bisymmetric triples of the form $(\mathfrak{so}_{2n}, \mathfrak{u}_n, \mathfrak{l})$ and $(\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2n-2p}, \mathfrak{l})$, $1 \le p \le n-1$.
A root system for $\mathfrak{g}$ is

$$\mathcal{R} = \{\pm e_i \pm e_j : 1 \le i < j \le n\} \tag{A.9}$$

and the length of any root is

$$|\alpha|^2 = \frac{1}{2(n-1)}. \tag{A.10}$$

Symmetric Pair A.3. $(\mathfrak{so}_{2n}, \mathfrak{u}_n)$.

$\mathfrak{k}$	$\mathcal{R}_\mathfrak{k}$	γ
$\mathfrak{u}_n$	$\{\pm(e_i - e_j) : 1 \le i < j \le n\}$	$\frac{n}{2(n-1)}$

$$\mathcal{R}_\mathfrak{n} = \{\pm(e_i + e_j) : 1 \le i < j \le n\}$$

Symmetric Pair A.4. $(\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2n-2p})$, $p = 1, \dots, n-1$.

$\mathfrak{k}_i$	$\mathcal{R}_{\mathfrak{k}_i}$	γ_i
$\mathfrak{so}_{2p}$	$\{\pm e_i \pm e_j : 1 \le i < j \le p\}$	$\frac{p-1}{n-1}$, $p \ge 2$
$\mathfrak{so}_{2n-2p}$	$\{\pm e_i \pm e_j : p+1 \le i < j \le n\}$	$\frac{n-p-1}{n-1}$, $p \le n-2$

$$\mathcal{R}_\mathfrak{n} = \{\pm e_i \pm e_j : 1 \le i \le p,\ p+1 \le j \le n\}$$

Bisymmetric Triple A.9. $(\mathfrak{so}_{2n}, \mathfrak{u}_n, \mathfrak{u}_p \oplus \mathfrak{u}_{n-p})$, $0 \le p \le n-1$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_\mathfrak{p} = \{\pm(e_i - e_j) : 1 \le i \le p,\ p+1 \le j \le n\}\\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \mathfrak{n}^3\\
&\mathcal{R}_{\mathfrak{n}^1} = \{\pm(e_i + e_j) : 1 \le i < j \le p\},\\
&\mathcal{R}_{\mathfrak{n}^2} = \{\pm(e_i + e_j) : p+1 \le i < j \le n,\},\\
&\mathcal{R}_{\mathfrak{n}^3} = \{\pm(e_i + e_j) : 1 \le i \le p,\ p+1 \le j \le n\}
\end{aligned}$$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_\mathfrak{p}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha'ss$	$\|\alpha\|^2$	b^ϕ
$\mathfrak{n}^1$	$e_1 + e_p$	$e_1 - e_i$, $p+1 \le i \le n$ $e_p - e_i$, $p+1 \le i \le n$	ϕ, $\phi - \alpha$	1	$n-p$ $n-p$	$\frac{1}{2(n-1)}$	$\frac{n-p}{2(n-1)}$
$\mathfrak{n}^2$	$e_{p+1} + e_n$	$e_i - e_{p+1}$, $1 \le i \le p$ $e_i - e_n$, $1 \le i \le p$	ϕ, $\phi + \alpha$	1	p p	$\frac{1}{2(n-1)}$	$\frac{p}{2(n-1)}$
$\mathfrak{n}^2$	$e_1 + e_n$	$e_1 - e_i$, $p+1 \le i \le n-1$ $e_i - e_n$, $2 \le i \le p$	ϕ, $\phi - \alpha$ ϕ, $\phi + \alpha$	1	$n-p-1$ $p-1$	$\frac{1}{2(n-1)}$	$\frac{n-2}{4(n-1)}$

Bisymmetric Triple A.10. $(\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2l} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2(n-p)})$, $1 \leq p \leq n-1$, $1 \leq l \leq p-1$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}} = \{\pm e_i \pm e_j : 1 \leq i \leq l,\, l+1 \leq j \leq p\}\\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2\\
&\mathcal{R}_{\mathfrak{n}^1} = \{\pm e_i \pm e_j : 1 \leq i \leq l,\, p+1 \leq j \leq n\}\\
&\mathcal{R}_{\mathfrak{n}^2} = \{\pm e_i \pm e_j : l+1 \leq i \leq p,\, p+1 \leq j \leq n\}
\end{aligned}$$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha's$	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$e_1 + e_n$	$e_1 \pm e_i$, $l+1 \leq i \leq p$	ϕ, $\phi - \alpha$	1	$2(p-l)$	$\frac{1}{2(n-1)}$	$\frac{p-l}{2(n-1)}$
$\mathfrak{n}^2$	$e_p + e_n$	$\pm e_i + e_p$, $1 \leq i \leq l$	ϕ, $\phi + \alpha$	1	l	$\frac{1}{2(n-1)}$	$\frac{l}{2(n-1)}$

Bisymmetric Triple A.11. $(\mathfrak{so}_{2n}, \mathfrak{s}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2s} \oplus \mathfrak{so}_{2(n-p-s)})$, $1 \leq p \leq n-1$, $1 \leq s \leq n-p-1$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}} = \{\pm e_i \pm e_j : p+1 \leq i \leq p+s,\, p+s+1 \leq j \leq n\}\\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2\\
&\mathcal{R}_{\mathfrak{n}^1} = \{\pm e_i \pm e_j : 1 \leq i \leq p,\, p+1 \leq j \leq p+s\}\\
&\mathcal{R}_{\mathfrak{n}^2} = \{\pm e_i \pm e_j : 1 \leq i \leq p,\, p+s+1 \leq j \leq n\}
\end{aligned}$$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha's$	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$e_1 + e_{p+1}$	$e_{p+1} \pm e_i$, $p+s+1 \leq i \leq n$	ϕ, $\phi - \alpha$	1	$2(n-p-s)$	$\frac{1}{2(n-1)}$	$\frac{n-p-s}{2(n-1)}$
$\mathfrak{n}^2$	$e_1 + e_n$	$\pm e_i + e_n$, $p+1 \leq i \leq p+s$	ϕ, $\phi - \alpha$	1	$2s$	$\frac{1}{2(n-1)}$	$\frac{s}{2(n-1)}$

Bisymmetric Triple A.12. $(\mathfrak{so}_{2n}, \mathfrak{s}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2l} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2s} \oplus \mathfrak{so}_{2(n-p-s)})$, $1 \leq p \leq n-1$, $1 \leq l \leq p-1$, $1 \leq s \leq n-p-1$. *(Type II)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}_1} = \{\pm e_i \pm e_j : 1 \leq i \leq l,\, l+1 \leq j \leq p\}\\
&\mathcal{R}_{\mathfrak{p}_2} = \{\pm e_i \pm e_j : p+1 \leq i \leq p+s,\, p+s+1 \leq j \leq n\}\\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \mathfrak{n}^3 \oplus \mathfrak{n}^4
\end{aligned}$$

$\mathcal{R}_{\mathfrak{n}^i}$	b_1^ϕ	b_2^ϕ
$\{\pm e_i \pm e_j : 1 \leq i \leq l,\, p+1 \leq j \leq p+s\}$	$\frac{p-l}{2(n-1)}$	$\frac{n-p-s}{2(n-1)}$
$\{\pm e_i \pm e_j : 1 \leq i \leq 1,\, p+s+1 \leq j \leq n\}$	$\frac{p-l}{2(n-1)}$	$\frac{s}{2(n-1)}$
$\{\pm e_i \pm e_j : l+1 \leq i \leq p,\, p+1 \leq j \leq p+s\}$	$\frac{l}{2(n-1)}$	$\frac{n-p-s}{2(n-1)}$
$\{\pm e_i \pm e_j : l+1 \leq i \leq p,\, p+s+1 \leq j \leq n\}$	$\frac{l}{2(n-1)}$	$\frac{s}{2(n-1)}$

Bisymmetric Triple A.13. $(\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{u}_p \oplus \mathfrak{so}_{2(n-p)})$, $1 \leq p \leq n-1$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i + e_j) : 1 \leq i < j \leq p\}\\
&\mathfrak{n} \textit{ irreducible } Ad\, L\textit{-module}
\end{aligned}$$

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha's$	$\lvert\alpha\rvert^2$	b^ϕ
$e_1 + e_n$	$e_1 + e_i$, $2 \leq i \leq p$	ϕ, $\phi - \alpha$	1	$p-1$	$\frac{1}{2(n-1)}$	$\frac{p-1}{4(n-1)}$

Bisymmetric Triple A.14. $(\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2p} \oplus \mathfrak{u}_{n-p})$, $1 \leq p \leq n-1$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i + e_j) : p+1 \leq i < j \leq n\}$
$\mathfrak{n}$ *irreducible* $Ad\, L$*-module*

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' s$	$\lvert\alpha\rvert^2$	b^ϕ
$e_1 + e_n$	$e_i + e_n$, $p+1 \leq i \leq n-1$	ϕ, $\phi - \alpha$	1	$n-p-1$	$\frac{1}{2(n-1)}$	$\frac{n-p-1}{4(n-1)}$

Bisymmetric Triple A.15. $(\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{u}_p \oplus \mathfrak{u}_{n-p})$, $1 \leq p \leq n-1$. *(Type II)*

$\mathcal{R}_{\mathfrak{p}_1} = \{\pm(e_i + e_j) : 1 \leq i < j \leq p\}$
$\mathcal{R}_{\mathfrak{p}_2} = \{\pm(e_i + e_j) : p+1 \leq i < j \leq n\}$
$\mathfrak{n}$ *irreducible* $Ad\, L$*-module*

b_1^ϕ	b_2^ϕ
$\frac{p-1}{4(n-1)}$	$\frac{n-p-1}{4(n-1)}$

Bisymmetric Triple A.16. $(\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2l} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{u}_{n-p})$, $1 \leq p \leq n-1$, $1 \leq l \leq p-1$. *(Type II)*

$\mathcal{R}_{\mathfrak{p}_1} = \{\pm e_i \pm e_j : 1 \leq i \leq l,\, l+1 \leq j \leq p\}$
$\mathcal{R}_{\mathfrak{p}_2} = \mathcal{R}_{\mathfrak{p}_2} = \{\pm(e_i + e_j) : p+1 \leq i < j \leq n\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2$

$\mathcal{R}_{\mathfrak{n}^i}$	b_1^ϕ	b_2^ϕ
$\{\pm e_i \pm e_j : 1 \leq i \leq l,\, p+1 \leq j \leq n\}$	$\frac{p-l}{2(n-1)}$	$\frac{n-p-1}{4(n-1)}$
$\{\pm e_i \pm e_j : l+1 \leq i \leq p,\, p+1 \leq j \leq n\}$	$\frac{l}{2(n-1)}$	$\frac{n-p-1}{4(n-1)}$

A.4 C_n

We consider the bisymmetric triples of the form $(\mathfrak{sp}_n, \mathfrak{u}_n, \mathfrak{l})$ and $(\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{l})$, for $1 \le p \le n-2$. The root system for $\mathfrak{g} = \mathfrak{sp}_n$ is

$$\mathcal{R} = \{\pm 2e_i : 1 \le i \le n;\ \pm e_i \pm e_j : 1 \le i < j \le n\}. \tag{A.11}$$

In $\mathfrak{g}$ there are two root lengths:

$$|\alpha|^2 = \begin{cases} \frac{1}{n+1}, & \alpha = \pm 2e_i \\ \frac{1}{2(n+1)}, & \alpha = \pm e_i \pm e_j \end{cases}. \tag{A.12}$$

Symmetric Pair A.5. $(\mathfrak{sp}_n, \mathfrak{u}_n)$.

$\mathfrak{k}$	$\mathcal{R}_{\mathfrak{k}}$	γ
$\mathfrak{u}_n$	$\{\pm(e_i - e_j) : 1 \le i < j \le n\}$	$\frac{n}{2(n+1)}$

$$\mathcal{R}_{\mathfrak{n}} = \{\pm 2e_i,\ \pm(e_i + e_j) : 1 \le i < j \le n\}$$

Symmetric Pair A.6. $(\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p})$, $p = 1, \dots, n-2$.

$\mathfrak{k}_i$	$\mathcal{R}_{\mathfrak{k}_i}$	γ_i
$\mathfrak{sp}_p$	$\{\pm 2e_i : 1 \le i \le p;\ \pm e_i \pm e_j : 1 \le i < j \le p\}$	$\frac{p+1}{n+1}$
$\mathfrak{sp}_{n-p}$	$\{\pm 2e_i : p+1 \le i \le n;\ \pm e_i \pm e_j : p+1 \le i < j \le n\}$	$\frac{n-p+1}{n+1}$

$$\mathcal{R}_{\mathfrak{n}} = \{\pm e_i \pm e_j : 1 \le i \le p,\ p+1 \le j \le n\}$$

Bisymmetric Triple A.17. $(\mathfrak{sp}_n, \mathfrak{u}_n, \mathfrak{u}_p \oplus \mathfrak{u}_{n-p})$, $1 \le p \le n-1$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i - e_j) : 1 \le i \le p,\ p+1 \le j \le n\} \\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \mathfrak{n}^3 \\
&\mathcal{R}_{\mathfrak{n}^1} = \{\pm 2e_i,\ \pm(e_i + e_j) : 1 \le i < j \le p\}, \\
&\mathcal{R}_{\mathfrak{n}^2} = \{\pm 2e_i,\ \pm(e_i + e_j) : p+1 \le i < j \le n,\}, \\
&\mathcal{R}_{\mathfrak{n}^3} = \{\pm(e_i + e_j) : 1 \le i \le p,\ p+1 \le j \le n\}
\end{aligned}$$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* α*'s*	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$2e_1$	$e_1 - e_i$, $p+1 \le i \le n$	$\phi,\ \phi - \alpha$	1	$n-p$	$\frac{1}{2(n+1)}$	$\frac{n-p}{4(n+1)}$
$\mathfrak{n}^2$	$2e_n$	$e_i - e_n$, $1 \le i \le p$	$\phi,\ \phi + \alpha$	1	p	$\frac{1}{2(n+1)}$	$\frac{p}{4(n+1)}$
		$e_i - e_n$, $2 \le i \le p$	$\phi,\ \phi + \alpha$	1	$p-1$		
$\mathfrak{n}^3$	$e_1 + e_n$	$e_1 - e_i$, $p+1 \le i \le n-1$	$\phi,\ \phi - \alpha$	1	$n-p-1$	$\frac{1}{2(n+1)}$	$\frac{n+2}{4(n+1)}$
		$e_1 - e_n$	$\phi,\ \phi \pm \alpha$	4	1		

Bisymmetric Triple A.18. $(\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{sp}_l \oplus \mathfrak{sp}_{p-l} \oplus \mathfrak{sp}_{n-p})$, $1 \le p \le n-1$, $1 \le l \le p-1$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}} = \{\pm e_i \pm e_j : 1 \le i \le l,\ l+1 \le j \le p\}\\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2\\
&\mathcal{R}_{\mathfrak{n}^1} = \{\pm e_i \pm e_j : 1 \le i \le l,\ p+1 \le j \le n\}\\
&\mathcal{R}_{\mathfrak{n}^2} = \{\pm e_i \pm e_j : l+1 \le i \le p,\ p+1 \le j \le n\}
\end{aligned}$$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}^+_{\mathfrak{p}}$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* α'*s*	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$e_1 + e_n$	$e_1 \pm e_i,\ l+1 \le i \le p$	$\phi,\ \phi-\alpha$	1	$p-l$	$\frac{1}{2(n+1)}$	$\frac{p-l}{4(n+1)}$
$\mathfrak{n}^2$	$e_p + e_n$	$\pm e_i + e_p,\ 1 \le i \le l$	$\phi,\ \phi-\alpha$	1	l	$\frac{1}{2(n+1)}$	$\frac{l}{4(n+1)}$

Bisymmetric Triple A.19. $(\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{sp}_p \oplus \mathfrak{sp}_s \oplus \mathfrak{sp}_{n-p-s})$, $1 \le p \le n-1$, $1 \le s \le n-p-1$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}} = \{\pm e_i \pm e_j : p+1 \le i \le p+s,\ p+s+1 \le j \le n\}\\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2\\
&\mathcal{R}_{\mathfrak{n}^1} = \{\pm e_i \pm e_j : 1 \le i \le p,\ p+1 \le j \le p+s\}\\
&\mathcal{R}_{\mathfrak{n}^2} = \{\pm e_i \pm e_j : l \le i \le p,\ p+s+1 \le j \le n\}
\end{aligned}$$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}^+_{\mathfrak{p}}$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* α'*s*	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$e_1 + e_{p+1}$	$e_{p+1} \pm e_i,\ p+s+1 \le i \le n$	$\phi,\ \phi-\alpha$	1	$n-p-s$	$\frac{1}{2(n+1)}$	$\frac{n-p-s}{4(n+1)}$
$\mathfrak{n}^2$	$e_p + e_n$	$\pm e_i + e_n,\ p+1 \le i \le p+s$	$\phi,\ \phi-\alpha$	1	s	$\frac{1}{2(n+1)}$	$\frac{s}{4(n+1)}$

Bisymmetric Triple A.20. $(\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{sp}_l \oplus \mathfrak{sp}_{p-l} \oplus \mathfrak{sp}_s \oplus \mathfrak{sp}_{n-p-s})$, $1 \le p \le n-1$, $1 \le l \le p-1$, $1 \le s \le n-p-1$. *(Type II)*

$$\mathcal{R}_{\mathfrak{p}_1} = \{\pm e_i \pm e_j : 1 \le i \le l,\ l+1 \le j \le p\}$$

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}_2} = \{\pm e_i \pm e_j : p+1 \le i \le p+s,\ p+s+1 \le j \le n\}\\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \mathfrak{n}^3 \oplus \mathfrak{n}^4
\end{aligned}$$

$\mathcal{R}_{\mathfrak{n}^i}$	b_1^ϕ	b_2^ϕ
$\{\pm e_i \pm e_j : 1 \le i \le l,\ p+1 \le j \le p+s\}$	$\frac{p-l}{4(n+1)}$	$\frac{n-p-s}{4(n+1)}$
$\{\pm e_i \pm e_j : 1 \le i \le l,\ p+s+1 \le j \le n\}$	$\frac{p-l}{4(n+1)}$	$\frac{s}{4(n+1)}$
$\{\pm e_i \pm e_j : l+1 \le i \le p,\ p+1 \le j \le p+s\}$	$\frac{l}{4(n+1)}$	$\frac{n-p-s}{4(n+1)}$
$\{\pm e_i \pm e_j : l+1 \le i \le p,\ p+s+1 \le j \le n\}$	$\frac{l}{4(n+1)}$	$\frac{s}{4(n+1)}$

Bisymmetric Triple A.21. $(\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{u}_p \oplus \mathfrak{sp}_{n-p})$, $1 \le p \le n-1$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}} = \{\pm 2e_i : 1 \le i \le p;\ \pm(e_i + e_j) : 1 \le i < j \le p\}\\
&\mathfrak{n}\ \textit{irreducible}\ Ad\,L\textit{-module}
\end{aligned}$$

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}^+_{\mathfrak{p}}$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* α'*s*	$\lvert\alpha\rvert^2$	b^ϕ
$e_1 + e_n$	$2e_1$ $e_1 + e_i,\ 2 \le i \le p$	$\phi,\ \phi-\alpha$	1	1 $p-1$	$\frac{1}{n+1}$ $\frac{1}{2(n+1)}$	$\frac{p+1}{4(n+1)}$

Bisymmetric Triple A.22. $(\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{sp}_p \oplus \mathfrak{u}_{n-p})$, $1 \le p \le n-1$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm 2e_i : p+1 \le i \le n;\ \pm(e_i+e_j) : p+1 \le i < j \le n\}$
$\mathfrak{n}$ *irreducible* $Ad\,L$*-module*

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' s$	$\lvert\alpha\rvert^2$	b^ϕ
$e_1 + e_n$	$2e_n$ $e_i + e_n$, $p+1 \le i \le n-1$	ϕ, $\phi - \alpha$	1	1 $n-p-1$	$\frac{1}{n+1}$ $\frac{1}{2(n+1)}$	$\frac{n-p+1}{4(n+1)}$

Bisymmetric Triple A.23. $(\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{u}_p \oplus \mathfrak{u}_{n-p})$, $1 \le p \le n-1$. *(Type II)*

$\mathcal{R}_{\mathfrak{p}_1} = \{\pm 2e_i : 1 \le i \le p;\ \pm(e_i+e_j) : 1 \le i < j \le p\}$
$\mathcal{R}_{\mathfrak{p}_2} = \{\pm 2e_i : p+1 \le i \le n;\ \pm(e_i+e_j) : p+1 \le i < j \le n\}$
$\mathfrak{n}$ *irreducible* $Ad\,L$*-module*

b_1^ϕ	b_2^ϕ
$\frac{p+1}{4(n+1)}$	$\frac{n-p+1}{4(n+1)}$

Bisymmetric Triple A.24. $(\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{sp}_l \oplus \mathfrak{sp}_{p-l} \oplus \mathfrak{u}_{n-p})$, $1 \le p \le n-1$ *and* $1 \le l \le p-1$. *(Type II)*

$\mathcal{R}_{\mathfrak{p}_1} = \{\pm e_i \pm e_j : 1 \le i \le l,\ l+1 \le j \le p\}$
$\mathcal{R}_{\mathfrak{p}_2} = \{\pm 2e_i : p+1 \le i \le n;\ \pm(e_i+e_j) : p+1 \le i < j \le n\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2$

$\mathcal{R}_{\mathfrak{n}^i}$	b_1^ϕ	b_2^ϕ
$\{\pm e_i \pm e_j : 1 \le i \le l,\ p+1 \le j \le n\}$	$\frac{p-l}{4(n+1)}$	$\frac{n-p+1}{4(n+1)}$
$\{\pm e_i \pm e_j : l+1 \le i \le p,\ p+1 \le j \le n\}$	$\frac{l}{4(n+1)}$	$\frac{n-p+1}{4(n+1)}$

A.5 $\mathfrak{f}_4$

In this section we analyze the bisymmetric triples of the form $(\mathfrak{f}_4, \mathfrak{so}_9, \mathfrak{l})$ and $(\mathfrak{f}_4, \mathfrak{sp}_3 \oplus \mathfrak{su}_2, \mathfrak{l})$.

The root system for the simple Lie algebra $\mathfrak{f}_4$ is

$$\mathcal{R} = \{\pm e_i,\ \pm e_i \pm e_j, 1 \leq i < j \leq 4;\ \frac{1}{2}\sum_1^4 (-1)^{\nu_i} e_i\}, \tag{A.13}$$

where $e_1, \ldots, e_4$ is the canonical basis for $\mathbb{R}^4$ and the signs are chosen independently. In $\mathfrak{f}_4$, there are two root lengths. Roots of the form $\pm e_i$ and $\frac{1}{2}\sum_1^4 (-1)^{\nu_i} e_i$ are short, whereas those of the form $\pm e_i \pm e_j$ are long, and we have

$$|\alpha|^2 = \begin{cases} \frac{1}{18}, \ \alpha \text{ short} \\ \frac{1}{9}, \ \alpha \text{ long} \end{cases} . \tag{A.14}$$

Symmetric Pair A.7. $(\mathfrak{f}_4, \mathfrak{so}_9)$.

$\mathfrak{k}$	$\mathcal{R}_\mathfrak{k}$	γ
$\mathfrak{so}_9$	$\{\pm e_i,\ \pm e_i \pm e_j, 1 \leq i < j \leq 4\}$	$\frac{7}{9}$

$$\mathcal{R}_\mathfrak{n} = \{\frac{1}{2}\sum_1^4 (-1)^{\nu_i} e_i\}$$

Symmetric Pair A.8. $(\mathfrak{f}_4, \mathfrak{sp}_3 \oplus \mathfrak{su}_2)$.

$\mathfrak{k}_i$	$\mathcal{R}_{\mathfrak{k}_i}$	γ_i
$\mathfrak{sp}_3$	$< e_4, e_3 - e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4) >=$ $\{\pm e_3, \pm e_4, \pm e_3 \pm e_4,\ \pm(e_1 - e_2),\ \pm\frac{1}{2}(e_1 - e_2 \pm e_3 \pm e_4)\}$	$\frac{4}{9}$
$\mathfrak{su}_2$	$\{\pm(e_1 + e_2)\}$	$\frac{2}{9}$

$\mathcal{R}_\mathfrak{n} = \{\pm e_i,\ \pm e_i \pm e_j : i = 1, 2, j = 3, 4; \pm\frac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4)\}$, *with signs chosen independently.*

Bisymmetric Triple A.25. $(\mathfrak{f}_4, \mathfrak{so}_9, \mathfrak{so}_p \oplus \mathfrak{so}_{9-p})$, $p = 2l + 1$, $l = 0, 1, 2, 3$. *(Type I)*

$\mathcal{R}_\mathfrak{p} = \{\pm 2e_i : p + 1 \leq i \leq n;\ \pm(e_i + e_j) : p + 1 \leq i < j \leq n\}$
$C_\mathfrak{p}$ *scalar on* $\mathfrak{n}$

$\phi \in \mathcal{R}_\mathfrak{n}$	$\alpha \in \mathcal{R}_\mathfrak{p}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* α*'s*	$\|\alpha\|^2$	b^ϕ
$\frac{1}{2}\sum_1^4 e_i$	$e_i,\ l + 1 \leq i \leq 4$ $e_i + e_j,\ 1 \leq i \leq l, l + 1 \leq j \leq 4$	$\phi,\ \phi - \alpha$	1	$4 - l$ $l(4 - l)$	$\frac{1}{18}$ $\frac{1}{9}$	$\frac{p(9-p)}{72}$

Bisymmetric Triple A.26. $(\mathfrak{f}_4, \mathfrak{sp}_3 \oplus \mathfrak{su}_2, \mathfrak{sp}_3 \oplus \mathbb{R})$. *(Type I)*

$$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_1+e_2)\}$$
$$C_{\mathfrak{p}} \text{ scalar on } \mathfrak{n}$$

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* α'*s*	$\vert\alpha\vert^2$	b^ϕ
e_1	e_1+e_2	$\phi,\, \phi-\alpha$	1	1	$\frac{1}{9}$	$\frac{1}{18}$

Bisymmetric Triple A.27. $(\mathfrak{f}_4, \mathfrak{sp}_3 \oplus \mathfrak{su}_2, \mathfrak{u}_3 \oplus \mathfrak{su}_2)$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm e_3,\, \pm e_3 \pm e_4,\, \pm(e_1-e_2),\, \pm\frac{1}{2}(e_1-e_2+e_3\pm e_4)\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2$
$\mathcal{R}_{\mathfrak{n}^1} = \{\pm(e_1+e_3),\, \pm(e_2-e_3)\}$
$\mathcal{R}_{\mathfrak{n}^2} = \{\pm e_1,\, \pm e_2,\, \pm(e_1-e_3),\, \pm(e_1+e_4), \pm(e_2\pm e_3),\, \pm(e_2\pm e_4),\, \pm\frac{1}{2}(e_1+e_2\pm e_3\pm e_4)\}$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* α'*s*	$\vert\alpha\vert^2$	b^ϕ
$\mathfrak{n}^1$	e_1+e_3	$e_3,\, \frac{1}{2} \pm (e_1-e_2+e_3\pm e_4)$	$\phi,\, \phi-\alpha$	1	3	$\frac{1}{18}$	$\frac{1}{4}$
		$e_3\pm e_4,\, e_1-e_2$			3	$\frac{1}{9}$	
$\mathfrak{n}^2$	e_1	e_3	$\phi,\, \phi\pm\alpha$	4	1	$\frac{1}{18}$	$\frac{2}{9}$
		e_1-e_2	$\phi,\, \phi-\alpha$	1	1	$\frac{1}{9}$	
		$\frac{1}{2} \pm (e_1-e_2+e_3\pm e_4)$	$\phi,\, \phi-\alpha$	1	2	$\frac{1}{18}$	

Bisymmetric Triple A.28. $(\mathfrak{f}_4, \mathfrak{sp}_3 \oplus \mathfrak{su}_2, \mathfrak{sp}_2 \oplus \mathfrak{su}_2 \oplus \mathfrak{su}_2)$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm e_3,\, \pm e_4,\, \pm\frac{1}{2}(e_1-e_2\pm(e_3+e_4)\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2$
$\mathcal{R}_{\mathfrak{n}^1} = \{\pm e_i \pm e_j,\, i=1,2,\, j=3,4;\, \pm\frac{1}{2}(e_1+e_2\pm(e_3+e_4))\}$
$\mathcal{R}_{\mathfrak{n}^2} = \{\pm e_1,\, \pm e_2,\, \pm\frac{1}{2}(e_1+e_2\pm(e_3-e_4))\}$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* α'*s*	$\vert\alpha\vert^2$	b^ϕ
$\mathfrak{n}^1$	$\frac{1}{2}(e_1+e_2+e_3+e_4)$	$\frac{1}{2}(e_1-e_2-e_3-e_4)$	$\phi,\, \phi+\alpha$	1	4	$\frac{1}{18}$	$\frac{1}{9}$
		all others	$\phi,\, \phi-\alpha$				
$\mathfrak{n}^2$	e_1	$e_3,\, e_4$	$\phi,\, \phi\pm\alpha$	4	2	$\frac{1}{18}$	$\frac{5}{18}$
		$\frac{1}{2}(e_1-e_2\pm(e_3+e_4))$	$\phi,\, \phi-\alpha$	1	2		

Bisymmetric Triple A.29. $(\mathfrak{f}_4, \mathfrak{sp}_3 \oplus \mathfrak{su}_2, \mathfrak{u}_3 \oplus \mathbb{R})$. *(Type II)*

$\mathcal{R}_{\mathfrak{p}_1} = \{\pm e_3,\, \pm e_3\pm e_4,\, \pm(e_1-e_2),\, \pm\frac{1}{2}(e_1-e_2+e_3\pm e_4)\}$
$\mathcal{R}_{\mathfrak{p}_2} = \{\pm(e_1+e_2)\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2$

$\mathcal{R}_{\mathfrak{n}^i}$	b_1^ϕ	b_2^ϕ
$\{\pm(e_1+e_3),\, \pm(e_2-e_3)\}$	$\frac{1}{4}$	$\frac{1}{18}$
$\{\pm e_1,\, \pm e_2,\, \pm(e_1-e_3),\, \pm(e_1+e_4), \pm(e_2\pm e_3),\, \pm(e_2\pm e_4),\, \pm\frac{1}{2}(e_1+e_2\pm e_3\pm e_4)\}$	$\frac{2}{9}$	$\frac{1}{18}$

Bisymmetric Triple A.30. $(\mathfrak{f}_4, \mathfrak{sp}_3 \oplus \mathfrak{su}_2, \mathfrak{sp}_2 \oplus \mathfrak{su}_2 \oplus \mathbb{R})$. *(Type II)*

$\mathcal{R}_{\mathfrak{p}_1} = \{\pm e_3,\ \pm e_4,\ \pm\frac{1}{2}(e_1 - e_2 \pm (e_3 + e_4)\}\mathcal{R}_{\mathfrak{p}_2} = \{\pm(e_1 + e_2)\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2$

$\mathcal{R}_{\mathfrak{n}^i}$	b_1^ϕ	b_2^ϕ
$\{\pm e_i \pm e_j,\ i = 1, 2,\ j = 3, 4;\ \pm\frac{1}{2}(e_1 + e_2 \pm (e_3 + e_4))\}$	$\frac{1}{9}$	$\frac{1}{18}$
$\{\pm e_1,\ \pm e_2,\ \pm\frac{1}{2}(e_1 + e_2 \pm (e_3 - e_4))\}$	$\frac{5}{18}$	$\frac{1}{18}$

A.6 $\mathfrak{g}_2$

In this section we analyze all bisymmetric triples $(\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2, \mathfrak{l})$. The symmetric pair $(\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2)$ corresponds to a flag manifold of G_2 and thus is obtained by a painted Dynkin diagram. We observe that each factor $\mathfrak{su}_2$ corresponds to a different root length. We set that $\mathfrak{k}_1 = \mathfrak{su}_2$ corresponds to a long root and $\mathfrak{k}_2 = \mathfrak{su}_2$ corresponds to a short root. If we consider the root system of $\mathfrak{g}_2$,

$$\begin{aligned}\mathcal{R} = \ & \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3), \\ & \pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\},\end{aligned}$$

we can choose $\mathcal{R}_{\mathfrak{k}_1} = \{\pm(2e_1 - e_2 - e_3)\}$ and $\mathcal{R}_{\mathfrak{k}_2} = \{\pm(e_2 - e_3)\}$, the orthogonal of $\mathcal{R}_{\mathfrak{k}_1}$.

In $\mathfrak{g}_2$ the length of a root is given by

$$|\alpha|^2 = \begin{cases} \frac{1}{4}, & \alpha \text{ long} \\ \frac{1}{12}, & \alpha \text{ short} \end{cases}, \tag{A.15}$$

where roots of the form $e_a - e_b$ are short and those of the form $2e_a - e_b - e_c$ are long.

Symmetric Pair A.9. $(\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2)$.

$\mathfrak{k}_i$	$\mathcal{R}_{\mathfrak{k}_i}$	γ_i
$\mathfrak{su}_2$	$\{\pm(2e_1 - e_2 - e_3)\}$	$\frac{1}{2}$
$\mathfrak{su}_2$	$\{\pm(e_2 - e_3)\}$	$\frac{1}{6}$

$$\mathcal{R}_{\mathfrak{n}} = \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\}$$

Bisymmetric Triple A.31. $(\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2, \mathbb{R} \oplus \mathfrak{su}_2)$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm(2e_1 - e_2 - e_3)\}$
$C_{\mathfrak{p}}$ *scalar on* $\mathfrak{n}$

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* α'*s*	$\|\alpha\|^2$	b^ϕ
$e_1 - e_2$	$2e_1 - e_2 - e_3$	$\phi, \phi - \alpha$	1	1	$\frac{1}{4}$	$\frac{1}{8}$

Bisymmetric Triple A.32. $(\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2, \mathfrak{su}_2 \oplus \mathbb{R})$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_2 - e_3)\}$
$C_{\mathfrak{p}}$ *scalar on* $\mathfrak{n}$

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* α'*s*	$\|\alpha\|^2$	b^ϕ
$e_1 - e_2$	$e_2 - e_3$	$\phi, \phi \pm \alpha$	4	1	$\frac{1}{12}$	$\frac{1}{6}$

Bisymmetric Triple A.33. $(\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2, \mathbb{R} \oplus \mathbb{R})$. *(Type II)*

$$\mathcal{R}_{\mathfrak{p}_1} = \{\pm(2e_1 - e_2 - e_3)\}$$
$$\mathcal{R}_{\mathfrak{p}_2} = \{\pm(e_2 - e_3)\}$$
$$C_{\mathfrak{p}_i} \textit{ scalar on } \mathfrak{n},\ i = 1, 2$$

b_1^ϕ	b_2^ϕ
$\frac{1}{8}$	$\frac{1}{6}$

A.7 $\mathfrak{e}_8$

In this section we consider the bisymmetric triples of the form $(\mathfrak{e}_8, \mathfrak{so}_{16}, \mathfrak{l})$ and $(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2, \mathfrak{l})$. The root system for $\mathfrak{e}_8$ is

$$\mathcal{R} = \{\pm e_i \pm e_j, 1 \le i < j \le 8;\ \pm\frac{1}{2}\sum_1^8 (-1)^{\nu_i} e_i, \sum_1^8 \nu_i \text{ even } \}, \tag{A.16}$$

where $e_1, \ldots, e_8$ is the canonical basis for $\mathbb{R}^8$. In $\mathfrak{e}_8$ there is only one root length which is

$$|\alpha|^2 = \frac{1}{30}. \tag{A.17}$$

Lemma A.1. *Let $\phi = \frac{1}{2}\sum_1^8 (-1)^{\nu_i} e_i \in \mathcal{R}$.*
(i) Let $\alpha = \frac{1}{2}\sum_1^8 (-1)^{\mu_i} e_i \in \mathcal{R}$. The string $\phi + n\alpha$ is either singular, ϕ, $\phi + \alpha$ or ϕ, $\phi - \alpha$. So either $d_{\alpha\phi} = 0$ or 1, respectively.
We have that $\phi + \alpha$ is a root if and only if $\nu_i = \mu_i$, for for two indices $i_1, i_2 \in \{1, \ldots, 8\}$ and in this case, $\phi + \alpha = (-1)^{\nu_{i_1}} e_{i_1} + (-1)^{\nu_{i_2}} e_{i_2}$.
$\phi - \alpha$ is a root if and only if $\nu_i \neq \mu_i$, for two indices $i_1, i_2 \in \{1, \ldots, 8\}$ and in this case, $\phi - \alpha = (-1)^{\nu_{i_1}} e_{i_1} + (-1)^{\nu_{i_2}} e_{i_2}$.
(ii) Let $\alpha' = (-1)^{\mu_j} e_j + (-1)^{\mu_k} e_k \in \mathcal{R}$, $1 \le j < k \le 8$. The string $\phi + n\alpha'$ is either singular or ϕ, $\phi - \alpha'$. $\phi - \alpha'$ is a root if and only if $\alpha' = (-1)^{\nu_j} e_j + (-1)^{\nu_k} e_k$, for $1 \le j < k \le 8$. In this case, $\phi - \alpha' = \frac{1}{2}\big(\sum_{\substack{i=1\\ i\neq j,k}}^8 (-1)^{\nu_i} e_i + (-1)^{\nu_j+1} e_j + (-1)^{\nu_k+1} e_k\big)$.[1]

Proof: Consider the roots $\phi = \frac{1}{2}\sum_1^8 (-1)^{\nu_i} e_i$ and $\alpha = \frac{1}{2}\sum_1^8 (-1)^{\mu_i} e_i$. Suppose that the string $\phi + n\alpha$ is not singular. Then, since $\phi + n\alpha$ is an uninterrupted string either $\phi + \alpha$ or $\phi - \alpha$ is a root. We have that

$$\phi + \alpha = \frac{1}{2}\sum_1^8 ((-1)^{\nu_i} + (-1)^{\mu_i}) e_i \text{ and } \phi - \alpha = \frac{1}{2}\sum_1^5 ((-1)^{\nu_i} - (-1)^{\mu_i}) e_i.$$

By observing the form of the roots in $\mathcal{R}$ given in (A.18) we conclude that $\phi + \alpha$ is a root if and only if $(-1)^{\nu_i} + (-1)^{\mu_i} = 0$, i.e., $\nu_i = \mu_i$, for two indices $i_1, i_2 \in \{1, \ldots, 8\}$. We observe that this case is possible since $\sum_1^8 \nu_i$ and $\sum_1^8 \mu_i$ are even with 8 even. We thus obtain $\phi + \alpha = (-1)^{\nu_{i_1}} e_{i_1} + (-1)^{\nu_{i_2}} e_{i_2}$. Clearly $\phi + 2\alpha$ is never a root. Similarly, $\phi - \alpha$ is a root if and only if $(-1)^{\nu_i} - (-1)^{\mu_i} \neq 0$, i.e.,$\nu_i \neq \mu_i$, for two indices $i_1, i_2 \in \{1, \ldots, 6\}$ and in this case, $\phi - \alpha = (-1)^{\nu_{i_1}} e_{i_1} + (-1)^{\nu_{i_2}} e_{i_2}$. The element $\phi - 2\alpha$ is never a root. Once again we observe that this case is possible .

Let $\alpha' = (-1)^{\mu_j} e_j + (-1)^{\mu_k} e_k \in \mathcal{R}$. We have

[1] We may choose $-\alpha$' in which case we obtain $\phi + \alpha'$ instead.

$$\phi - \alpha' = \frac{1}{2}\Big(\sum_{\substack{i=1\\ i\neq j,k}}^{8}(-1)^{\nu_i}e_i + ((-1)^{\nu_j} - 2(-1)^{\mu_j})e_j + ((-1)^{\nu_k} - 2(-1)^{\mu_k})e_k\Big).$$

This element is a root if and only if $\mu_j = \nu_j$ and $\mu_k = \nu_k$ and, in this case,

$$\phi - \alpha' = \frac{1}{2}\Big(\sum_{\substack{i=1\\ i\neq j,k}}^{8}(-1)^{\nu_i}e_i + (-1)^{\nu_j+1}e_j + (-1)^{\nu_k+1}e_k\Big).$$

We observe that $\phi - 2\alpha'$ is never a root.

$\square$

Symmetric Pair A.10. $(\mathfrak{e}_8, \mathfrak{so}_{16})$.

$\mathfrak{k}$	$\mathcal{R}_{\mathfrak{k}}$	γ
$\mathfrak{so}_{16}$	$\{\pm e_i \pm e_j, 1 \le i < j \le 8\}$	$\frac{7}{15}$

$$\mathcal{R}_{\mathfrak{n}} = \mathcal{R}_{\mathfrak{n}} = \{\pm\frac{1}{2}\sum_1^8(-1)^{\nu_i}e_i, \sum_1^8 \nu_i \ \textit{even}\ \}$$

Symmetric Pair A.11. $(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2)$.

$\mathfrak{k}_i$	$\mathcal{R}_{\mathfrak{k}_i}$	γ_i
$\mathfrak{e}_7$	$\{\pm e_i \pm e_j, 1 \le i < j \le 6;\ \pm(e_7 - e_8);\ \pm\frac{1}{2}(e_7 - e_8 + \sum_1^6(-1)^{\nu_i}e_i), \sum_1^6 \nu_i \ \textit{odd}\ \}$	$\frac{3}{5}$
$\mathfrak{su}_2$	$\{\pm(e_7 + e_8)\}$	$\frac{1}{15}$

$$\mathcal{R}_{\mathfrak{n}} = \{\pm e_i \pm e_j, 1 \le i \le 6,\ j = 7, 8;\ \pm\frac{1}{2}(e_7 + e_8 + \sum_1^6(-1)^{\nu_i}e_i), \sum_1^6 \nu_i \ \textit{even}\ \}$$

Bisymmetric Triple A.34. $(\mathfrak{e}_8, \mathfrak{so}_{16}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(8-p)})$, $1 \le p \le 4$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm e_i \pm e_j, 1 \le i \le p, p+1 \le j \le 8\}$
$C_{\mathfrak{p}}$ *scalar on* $\mathfrak{n}$

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* α'*s*	$\lvert\alpha\rvert^2$	b^ϕ
$\frac{1}{2}\sum_1^8 e_i$	$e_i + e_j$, $1 \le i \le p$, $p+1 \le j \le 8$	ϕ, $\phi - \alpha$	1	$p(8-p)$	$\frac{1}{30}$	$\frac{p(8-p)}{60}$

Bisymmetric Triple A.35. $(\mathfrak{e}_8, \mathfrak{so}_{16}, \mathfrak{u}_8)$. *(Type I)*

$$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i + e_j), 1 \le i \le 8\}$$
$$\mathfrak{n} = \oplus_{0,1,2}\mathfrak{n}^i$$
$$\mathcal{R}_{\mathfrak{n}^i} = \{\pm\tfrac{1}{2}\textstyle\sum_1^8(-1)^{\nu_j}e_j : 2i \ \textit{odd} \ \nu_j's\},\ i = 0, 1, 2$$

$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha's$	$\lvert\alpha\rvert^2$	b^ϕ
$\frac{1}{2}\sum_1^8(-1)^{\nu_j}e_j,\ 2i\ \textit{odd}\ \nu_j's$	$e_a + e_b,\ \textit{s.t.}\ \nu_a = \nu_b$	$\phi,\ \phi + (-1)^{\nu_a+1}\alpha$	1	$4i^2 - 16i + 28^{(*)}$	$\frac{1}{30}$	$\frac{i^2-4i+7}{15}$

(*) the number of possible pairs (a, b), where $a < b$ and $\nu_a = \nu_b$, since $2i$ ν_j's are odd and $8 - 2i$ ν_j's are even, is

$$\begin{aligned} & \binom{2i}{2} + \binom{8-2i}{2} \\ = \ & 4i^2 - 16i + 28. \end{aligned}$$

Bisymmetric Triple A.36. $(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2, \mathfrak{e}_7 \oplus \mathbb{R})$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_7 + e_8)\}$
$C_{\mathfrak{p}}$ *scalar on* $\mathfrak{n}$

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}^+_{\mathfrak{p}}$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha's$	$\lvert\alpha\rvert^2$	b^ϕ
$\frac{1}{2}(e_7 + e_8 + \sum_1^6 (-1)^{\nu_i} e_i)$, $\sum_1^6 \nu_i$ *even*	$e_7 + e_8$	$\phi, \phi - \alpha$	1	1	$\frac{1}{30}$	$\frac{1}{60}$

Bisymmetric Triple A.37. $(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2, \mathfrak{e}_6 \oplus \mathbb{R} \oplus \mathfrak{su}_2)$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm e_i \pm e_6, 1 \le i \le 5;\ \pm(e_7 - e_8);\ \pm\frac{1}{2}(e_8 - e_7 + e_6 + \sum_1^5 (-1)^{\nu_i} e_i), \sum_1^5 \nu_i \textit{ odd }\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \mathfrak{n}^3$
$\mathcal{R}_{\mathfrak{n}^1} = \{\pm e_i \pm e_j : 1 \le i \le 5,\ j = 7, 8\}$,
$\mathcal{R}_{\mathfrak{n}^2} = \{\pm(e_6 - e_7),\ \pm(e_6 + e_8),\ \pm\frac{1}{2}(e_8 + e_7 - e_6 + \sum_1^5 (-1)^{\nu_i} e_i), \sum_1^6 \nu_i \textit{ odd }\}$,
$\mathcal{R}_{\mathfrak{n}^3} = \{\pm(e_6 + e_7),\ \pm(e_6 - e_8),\ \pm\frac{1}{2}(e_8 + e_7 + e_6 + \sum_1^5 (-1)^{\nu_i} e_i), \sum_1^6 \nu_i \textit{ even }\}$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}^+_{\mathfrak{p}}$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha's$	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$e_1 + e_8$	$e_1 \pm e_6,\ e_7 - e_8$	$\phi, \phi - \alpha$	1	3	$\frac{1}{30}$	$\frac{11}{60}$
		$\frac{1}{2}(e_8 - e_7 + e_6 + e_1 + \sum_2^5 (-1)^{\nu_i} e_i)$, $\sum_2^5 \nu_i$ *even*			2^3		
$\mathfrak{n}^2$	$e_6 - e_8$	$e_i + e_6,\ i \le i \le 5$	$\phi, \phi - \alpha$	1	5	$\frac{1}{30}$	$\frac{11}{60}$
		$e_i - e_6,\ i \le i \le 5$	$\phi, \phi + \alpha$		5		
		$e_7 - e_8$	$\phi, \phi + \alpha$		1		
$\mathfrak{n}^2$	$e_6 + e_8$	$e_i + e_6,\ i \le i \le 5$	$\phi, \phi - \alpha$	1	5	$\frac{1}{30}$	$\frac{9}{20}$
		$e_i - e_6,\ i \le i \le 5$	$\phi, \phi + \alpha$		5		
		$e_7 - e_8$	$\phi, \phi + \alpha$		1		
		$\frac{1}{2}(e_8 - e_7 + e_6 + \sum_1^5 (-1)^{\nu_j} e_j$, $\sum_1^5 \nu_j$ *odd*	$\phi, \phi - \alpha$		2^4		

Bisymmetric Triple A.38. $(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2, \mathfrak{e}_6 \oplus \mathbb{R} \oplus \mathbb{R})$. *(Type II)*

$\mathcal{R}_{\mathfrak{p}_1} = \{\pm(e_7 + e_8)\}$
$\mathcal{R}_{\mathfrak{p}_2} = \{\pm e_i \pm e_6, 1 \le i \le 5;\ \pm(e_7 - e_8);\ \pm\frac{1}{2}(e_8 - e_7 + e_6 + \sum_1^5 (-1)^{\nu_i} e_i), \sum_1^5 \nu_i \textit{ odd }\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \mathfrak{n}^3$

$\mathcal{R}_{\mathfrak{n}^i}$	b_1^ϕ	b_2^ϕ
$\{\pm e_i \pm e_j : 1 \le i \le 5,\ j = 7, 8\}$	$\frac{11}{60}$	$\frac{1}{60}$
$\{\pm(e_6 - e_7),\ \pm(e_6 + e_8),\ \pm\frac{1}{2}(e_8 + e_7 - e_6 + \sum_1^5 (-1)^{\nu_i} e_i), \sum_1^6 \nu_i \textit{ odd }\}$	$\frac{11}{60}$	$\frac{1}{60}$
$\{\pm(e_6 + e_7),\ \pm(e_6 - e_8),\ \pm\frac{1}{2}(e_8 + e_7 + e_6 + \sum_1^5 (-1)^{\nu_i} e_i), \sum_1^6 \nu_i \textit{ even }\}$	$\frac{9}{20}$	$\frac{1}{60}$

Bisymmetric Triple A.39. $(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2, \mathfrak{so}_{12} \oplus \mathfrak{su}_2 \oplus \mathfrak{su}_2)$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm\frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\nu_i} e_i), \sum_1^6 \nu_i \textit{ odd }\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2$
$\mathcal{R}_{\mathfrak{n}^1} = \{\pm e_i \pm e_j : 1 \le i \le,\ j = 7, 8\}$,
$\mathcal{R}_{\mathfrak{n}^2} = \{\pm\frac{1}{2}(e_8 + e_7 + \sum_1^6 (-1)^{\nu_i} e_i), \sum_1^6 \nu_i \textit{ even }\}$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha's$	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$e_1 + e_7$	$\frac{1}{2}(e_7 - e_8 + e_1 + \sum_2^6 (-1)^{\nu_i} e_i)$, $\sum_2^6 \nu_i$ *odd*	$\phi,\, \phi - \alpha$	1	2^4	$\frac{1}{30}$	$\frac{4}{15}$
$\mathfrak{n}^2$	$\frac{1}{2}\sum_1^8 e_j$	$\frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\nu_i} e_i)$, *1 even* ν_i	$\phi,\, \phi + \alpha$	1	6	$\frac{1}{30}$	$\frac{1}{5}$
		$\frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\nu_i} e_i)$, *1 odd* ν_i	$\phi,\, \phi - \alpha$		6		

Bisymmetric Triple A.40. $(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2, \mathfrak{so}_{12} \oplus \mathfrak{su}_2 \oplus \mathbb{R})$. *(Type II)*

$$\mathcal{R}_{\mathfrak{p}_1} = \{\pm\tfrac{1}{2}(e_7 - e_8 + \textstyle\sum_1^6 (-1)^{\nu_i} e_i), \sum_1^6 \nu_i \ \textit{odd}\ \}$$
$$\mathcal{R}_{\mathfrak{p}_2} = \mathcal{R}_{\mathfrak{k}_2} = \{\pm(e_7 + e_8)\}$$
$$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2$$

$\mathcal{R}_{\mathfrak{n}^i}$	b_1^ϕ	b_2^ϕ
$\{\pm e_i \pm e_j : 1 \leq i \leq,\, j = 7, 8\}$	$\frac{4}{15}$	$\frac{1}{60}$
$\{\pm\frac{1}{2}(e_8 + e_7 + \sum_1^6 (-1)^{\nu_i} e_i), \sum_1^6 \nu_i \ \textit{even}\ \}$	$\frac{1}{5}$	$\frac{1}{60}$

Bisymmetric Triple A.41. $(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2, \mathfrak{su}_8 \oplus \mathfrak{su}_2)$. *(Type I)*

$$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i + e_j) : 1 \leq i < j \leq 6;\ \pm\tfrac{1}{2}(e_7 - e_8 + \textstyle\sum_1^6 (-1)^{\nu_i} e_i),\ 3\,odd\,\nu_i' s\}$$
$\mathfrak{n}$ *irreducible Ad L-module*

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha's$	$\lvert\alpha\rvert^2$	b^ϕ
$e_1 + e_7$	$\frac{1}{2}(e_7 - e_8 + e_1 + \sum_2^6 (-1)^{\nu_i} e_i)$, *3 odd* $\nu_i' s$; $e_1 + e_j,\, j \in \{2, \ldots, 6\}$	$\phi,\, \phi - \alpha$	1	$\binom{5}{3}$, 5	$\frac{1}{30}$	$\frac{1}{4}$

Bisymmetric Triple A.42. $(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2, \mathfrak{su}_8 \oplus \mathbb{R})$. *(Type II)*

$$\mathcal{R}_{\mathfrak{p}_1} = \{\pm(e_i + e_j) : 1 \leq i < j \leq 6;\ \pm\tfrac{1}{2}(e_7 - e_8 + \textstyle\sum_1^6 (-1)^{\nu_i} e_i),\ 3\,odd\,\nu_i' s\}$$
$$\mathcal{R}_{\mathfrak{p}_2} = \{\pm(e_7 + e_8)\}$$
$\mathfrak{n}$ *irreducible Ad L-module*

b_1^ϕ	b_2^ϕ
$\frac{1}{4}$	$\frac{1}{60}$

A.8 $\mathfrak{e}_7$

In this Section we study the bisymmetric triples of the form $(\mathfrak{e}_7, \mathfrak{su}_8, \mathfrak{l})$, $(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{l})$ and $(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R}, \mathfrak{l})$.
The root system for $\mathfrak{g} = \mathfrak{e}_7$ is

$$\mathcal{R} = \{\pm e_i \pm e_j, 1 \leq i < j \leq 6;\ \pm(e_7 - e_8);\ \pm\frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\nu_i} e_i), \sum_1^6 \nu_i \text{ odd } \}, \tag{A.18}$$

where $e_1, \ldots, e_8$ is the canonical basis for $\mathbb{R}^8$. Throughout all the relations for the ν_i's are *mod* 2. In $\mathfrak{e}_7$, all the roots have the same length which is

$$|\alpha|^2 = \frac{1}{18}. \tag{A.19}$$

Lemma A.2. *Let* $\phi = \frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\nu_i} e_i) \in \mathcal{R}$.
(i) Let $\alpha = \frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\mu_i} e_i) \in \mathcal{R}$. *The string* $\phi + n\alpha$ *is either singular,* ϕ, $\phi + \alpha$ *or* ϕ, $\phi - \alpha$. *So either* $d_{\alpha\phi} = 0$ *or* 1, *respectively. We have that* $\phi + \alpha$ *is a root if and only if* $\nu_i \neq \mu_i$, *for every* $i = 1, \ldots, 6$ *and in this case,* $\phi + \alpha = e_8 - e_7$; $\phi - \alpha$ *is a root if and only if* $\nu_i \neq \mu_i$, *for two indices* $i_1, i_2 \in \{1, \ldots, 6\}$ *and in this case,* $\phi - \alpha = (-1)^{\nu_{i_1}} e_{i_1} + (-1)^{\nu_{i_2}} e_{i_2}$.
(ii) Let $\alpha' = (-1)^{\mu_j} e_j + (-1)^{\mu_k} e_k \in \mathcal{R}$, $1 \leq j < k \leq 6$. *The string* $\phi + n\alpha'$ *is either singular or* ϕ, $\phi - \alpha'$. $\phi - \alpha'$ *is a root if and only if* $\alpha' = (-1)^{\nu_j} e_j + (-1)^{\nu_k} e_k$, *for* $1 \leq j < k \leq 6$. *In this case,* $\phi - \alpha' = \frac{1}{2}\big(e_7 - e_8 + \sum_{\substack{i=1 \\ i \neq j,k}}^6 (-1)^{\nu_i} e_i + (-1)^{\nu_j + 1} e_j + (-1)^{\nu_k + 1} e_k\big)$.[2]

(iii) For $\alpha'' = e_7 - e_8 \in \mathcal{R}$, the string $\phi + n\alpha''$ is ϕ, $\phi - \alpha''$, with $\phi - \alpha'' = -\frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\nu_i + 1} e_i)$.
Proof: $\mathcal{R}$ is a subsystem of roots of the root system for $\mathfrak{e}_8$. Hence, we use Lemma A.1.
For (i) let us consider the roots $\phi = \frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\nu_i} e_i)$ and $\alpha = \frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\mu_i} e_i)$. Suppose that the string $\phi + n\alpha$ is not singular. Since $\nu_7 = \mu_7$ and $\nu_8 = \mu_8$ at least two indices satisfy $\nu_i = \mu_i$. Hence, if for every $i \in \{1, \ldots, 6\}$, $\nu_i \neq \mu_i$, then $\phi + \alpha = e_8 - e_7$ is a root; if there is $i \in \{1, \ldots, 6\}$, $\nu_i = \mu_i$, then we obtain a root $\phi - \alpha$ if and only if $\nu_i \neq \mu_i$ for precisely two indices $i_1, i_2 \in \{1, \ldots, 6\}$. In this case, $\phi - \alpha = (-1)^{\nu_{i_1}} e_{i_1} + (-1)^{\nu_{i_2}} e_{i_2}$. We observe that both conditions on indices in $\{1, \ldots, 6\}$ are possible since $\sum_i^6 \nu_i$ and $\sum_i^6 \mu_i$ are odd and 6 is even.
(ii) and (iii) follow directly from (ii) in Lemma A.1.
□

[2]We may choose $-\alpha$' in which case we obtain $\phi + \alpha'$ instead.

Symmetric Pair A.12. $(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2)$.

$\mathfrak{k}_i$	$\mathcal{R}_{\mathfrak{k}_i}$	γ_i
$\mathfrak{so}_{12}$	$\{\pm e_i \pm e_j, 1 \le i < j \le 6\}$	$\frac{5}{9}$
$\mathfrak{su}_2$	$\{\pm(e_7 - e_8)\}$	$\frac{1}{9}$

$$\mathcal{R}_{\mathfrak{n}} = \{\pm\frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\nu_i} e_i, \sum_1^6 \nu_i \ \textit{odd} \ \}$$

Symmetric Pair A.13. $(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R})$.

$\mathfrak{k}$	$\mathcal{R}_{\mathfrak{k}}$	γ
$\mathfrak{e}_6$	$\{\pm e_i \pm e_j : 1 \le i < j \le 5;\ \pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 (-1)^{\nu_i} e_i) : \sum_1^5 \nu_i \ \textit{is even}\}$	$\frac{2}{3}$

$$\mathcal{R}_{\mathfrak{n}} = \{\pm e_i \pm e_6 : 1 \le i \le 5;\ \pm(e_7 - e_8)\ \pm \frac{1}{2}(e_7 - e_8 + e_6 \sum_1^5 (-1)^{\nu_i} e_i), \sum_1^5 \nu_i \ \textit{odd} \ \}$$

Symmetric Pair A.14. $(\mathfrak{e}_7, \mathfrak{su}_8, \mathfrak{su}_p \oplus \mathfrak{su}_{8-p} \oplus \mathbb{R})$, $p = 1, \dots, 4$.

$\mathfrak{k}$	$\mathcal{R}_{\mathfrak{k}}$	γ
$\mathfrak{su}_8$	$\{\pm(e_i - e_j) :\ 1 \le i < j \le 6;\ \pm(e_7 - e_8); \pm\frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\nu_i} e_i) :\ 1\, \textit{or}\, 5\, \textit{odd}\, \nu_i' s\}$	$\frac{4}{9}$

$$\mathcal{R}_{\mathfrak{n}} = \{\pm(e_i + e_j) :\ 1 \le i < j \le 6;\ \pm\frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\nu_i} e_i) : 3\, \textit{odd}\, \nu_i' s\}$$

Bisymmetric Triple A.43. $(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{so}_{12} \oplus \mathbb{R})$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_7 - e_8)\}$
$C_{\mathfrak{p}}$ *scalar on* $\mathfrak{n}$

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' s$	$\lvert\alpha\rvert^2$	b^ϕ
$\frac{1}{2}(e_7 - e_8 - e_1 \sum_2^6 e_i)$	$e_7 - e_8$	$\phi,\ \phi - \alpha$	1	1	$\frac{1}{18}$	$\frac{1}{36}$

Bisymmetric Triple A.44. $(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{u}_6 \oplus \mathfrak{su}_2)$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i + e_j) : 1 \le i < j \le 6\}$
$\mathfrak{n} = \oplus_{0,1,2} \mathfrak{n}^i$
$\mathcal{R}_{\mathfrak{n}^i} = \{\pm\frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\nu_j} e_j :\ 2i + 1\, \nu_j' s \ \textit{are odd}\}$, $i = 0, 1, 2$

$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' s$	$\lvert\alpha\rvert^2$	b^ϕ
$\frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\nu_j} e_j$, $2i + 1\, \nu_j' s \ \textit{odd}$	$e_a + e_b$, $\nu_a = \nu_b$	$\phi,\ \phi - (-1)^{\nu_a} \alpha$	1	$4i^2 - 8i + 10^{(*)}$	$\frac{1}{18}$	$\frac{2i^2 - 4i + 5}{18}$

(*) the number of possible pairs (a, b), where $a \ne b$ and $\nu_a = \nu_b$, since $2i + 1$ ν_j's are odd, is

$$\binom{2i+1}{2} + \binom{6 - (2i+1)}{2} = 4i^2 - 8i + 10.$$

Bisymmetric Triple A.45. $(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{u}_6 \oplus \mathbb{R})$. *(Type II)*

$$\mathcal{R}_{\mathfrak{p}_1} = \{\pm(e_i + e_j) : 1 \le i < j \le 6\}$$
$$\mathcal{R}_{\mathfrak{p}_2} = \mathcal{R}_{\mathfrak{k}_2} = \{\pm(e_7 - e_8)\}$$
$$\mathfrak{n} = \oplus_{0,1,2}\mathfrak{n}^i$$

$\mathcal{R}_{\mathfrak{n}^i}$	b_1^ϕ	b_2^ϕ
$\{\pm\frac{1}{2}(e_7 - e_8 + \sum_1^6 (-1)^{\nu_j} e_j :\ 2i+1\ \nu_j' s \text{ are odd}\}^{(*)}$	$\frac{2i^2-4i+5}{18}$	$\frac{1}{36}$

(*) $i = 0, 1, 2$ as in A.44.

Bisymmetric Triple A.46. $(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{so}_p \oplus \mathfrak{so}_{12-p} \oplus \mathfrak{su}_2)$, $p = 2, 4, 6$. *(Type I)*

$$\mathcal{R}_{\mathfrak{p}} = \{\pm e_i \pm e_j : 1 \le i \le p/2,\ p/2 + 1 \le j \le 6\}$$
$C_{\mathfrak{p}}$ *scalar on* $\mathfrak{n}$

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' s$	$\lvert\alpha\rvert^2$	b^ϕ
$\frac{1}{2}(e_7 - e_8 \sum_1^6 e_i)$	$e_i - e_j$, $1 \le j \le 6$	ϕ, $\phi - \alpha$	1	$\frac{p(12-p)}{4}$	$\frac{1}{18}$	$\frac{p(12-p)}{144}$

Bisymmetric Triple A.47. $(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{so}_p \oplus \mathfrak{so}_{12-p} \oplus \mathbb{R})$, $p = 2, 4, 6$. *(Type II)*

$$\mathcal{R}_{\mathfrak{p}_1} = \{\pm e_i \pm e_j : 1 \le i \le p/2,\ p/2 + 1 \le j \le 6\}$$
$$\mathcal{R}_{\mathfrak{p}_2} = \{\pm(e_7 - e_8)\}$$
$C_{\mathfrak{p}_i}$ *scalar on* $\mathfrak{n}$, $i = 1, 2$

b_1^ϕ	b_2^ϕ
$\frac{1}{36}$	$\frac{p(12-p)}{144}$

Bisymmetric Triple A.48. $(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R}, \mathfrak{so}_{10} \oplus \mathbb{R} \oplus \mathbb{R})$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}} = \{\pm\tfrac{1}{2}(e_7 - e_8 - e_6 + \textstyle\sum_1^5 (-1)^{\nu_i} e_i, \sum_1^5 \nu_i \ \textit{even}\ \} \\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \mathfrak{n}^3 \\
&\mathcal{R}_{\mathfrak{n}^1} = \mathcal{R}_{\mathfrak{n}^1} = \{\pm e_i \pm e_6 : 1 \le i \le 5\}, \\
&\mathcal{R}_{\mathfrak{n}^2} = \{\pm\tfrac{1}{2}(e_7 - e_8 + e_6 + \textstyle\sum_1^5 (-1)^{\nu_i} e_i), \sum_1^5 \nu_i \, \textit{odd}\} \\
&\mathcal{R}_{\mathfrak{n}^3} = \{\pm(e_7 - e_8)\}
\end{aligned}$$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' s$	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$e_1 - e_6$	$\frac{1}{2}(e_7 - e_8 - e_6 + e_1 + \sum_2^5 (-1)^{\nu_i} e_i)$, $\sum_2^5 \nu_i$ *even*	ϕ, $\phi - \alpha$	1	2^3	$\frac{1}{18}$	$\frac{2}{9}$
$\mathfrak{n}^2$	$\frac{1}{2}(e_7 - e_8 + e_6 - e_5 + \sum_1^4 e_j)$	$\frac{1}{2}(e_7 - e_8 - e_6 + e_5 - \sum_1^4 e_i)$ $\frac{1}{2}(e_7 - e_8 - e_6 + \sum_1^5 e_i)$ $\frac{1}{2}(e_7 - e_8 - e_6 - e_5 + \sum_1^4 (-1)^{\nu_i} e_i)$, *one odd* ν_i	ϕ, $\phi + \alpha$ ϕ, $\phi - \alpha$ ϕ, $\phi - \alpha$	1	1 1 4	$\frac{1}{18}$	$\frac{1}{6}$
$\mathfrak{n}^3$	$e_7 - e_8$	*all*	ϕ, $\phi - \alpha$	1	2^4	$\frac{1}{18}$	$\frac{4}{9}$

Bisymmetric Triple A.49. $(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R}, \mathfrak{su}_6 \oplus \mathfrak{su}_2 \oplus \mathbb{R})$. *(Type I)*

$$\begin{aligned}
&\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i + e_j) : 1 \le i < j \le 5; \pm\tfrac{1}{2}(e_8 - e_7 - e_6 + \textstyle\sum_1^5 (-1)^{\nu_i} e_i), \ 2\ \nu_i' s\ \ \textit{odd}\} \\
&\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \\
&\mathcal{R}_{\mathfrak{n}^1} = \{\pm(e_i - e_6) : 1 \le i \le 5; \pm(e_7 - e_8); \pm\tfrac{1}{2}(e_8 - e_7 - e_6 + \textstyle\sum_1^5 (-1)^{\nu_i} e_i), \ 1\, \textit{or}\, 5\, \nu_i' s\, \textit{odd}\} \\
&\mathcal{R}_{\mathfrak{n}^2} = \{\pm(e_i + e_6) : 1 \le i \le 5; \pm\tfrac{1}{2}(e_8 - e_7 - e_6 + \textstyle\sum_1^5 (-1)^{\nu_i} e_i), 3\, \nu_i' s\, \textit{odd}\}
\end{aligned}$$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' s$	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$e_7 - e_8$	$\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 (-1)^{\nu_i} e_i)$, 2 *odd* $\nu_i' s$	ϕ, $\phi - \alpha$	1	$\binom{5}{2}$	$\frac{1}{18}$	$\frac{5}{18}$
$\mathfrak{n}^2$	$e_1 + e_6$	$e_1 - e_i$, $i \le i \le 5$ $\frac{1}{2}(e_8 - e_7 - e_6 - e_1 + \sum_2^5 (-1)^{\nu_i} e_i)$, 1 *odd* ν_i	ϕ, $\phi - \alpha$ ϕ, $\phi + \alpha$	1	4 4	$\frac{1}{18}$	$\frac{2}{9}$

Bisymmetric Triple A.50. $(\mathfrak{e}_7, \mathfrak{su}_8, \mathfrak{su}_p \oplus \mathfrak{su}_{8-p} \oplus \mathbb{R})$, $p = 1, \ldots, 4$. *(Type I)*

$$\mathcal{R}_{\mathfrak{l}} = \{\pm(e_i - e_j) : 1 \le i < j \le p \, or \, p+1 \le i < j \le 6;\ \pm(e_7 - e_8);\quad \pm\tfrac{1}{2}(e_7 - e_8 + \textstyle\sum_1^p e_i + \sum_{p+1}^6 (-1)^{\nu_i} e_i) : 1 \, odd \, \nu_i;$$
$$\pm\tfrac{1}{2}(e_7 - e_8 - \textstyle\sum_1^p e_i + \sum_{p+1}^6 (-1)^{\nu_i} e_i) : (5-p) \, odd \, \nu_i' s\}$$

$$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i - e_j) : 1 \le i \le p,\ p+1 \le j \le 6;\ \pm\tfrac{1}{2}(e_7 - e_8 + \textstyle\sum_1^p (-1)^{\nu_i} e_i + \sum_{p+1}^6 e_i) : 1 \, odd \, \nu_i;\ \pm\tfrac{1}{2}(e_7 - e_8 + \sum_1^p (-1)^{\nu_i} e_i - \sum_{p+1}^6 e_i) : (p-1) \, odd \, \nu_i' s\}$$

p=1

$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_1 - e_j) : 2 \le j \le 6; \pm\frac{1}{2}(e_7 - e_8 + e_1 - \sum_2^6 e_i);\ \pm\frac{1}{2}(e_7 - e_8 - e_1 + \sum_2^6 e_i)\}$
$\mathfrak{n}$ *irreducible Ad L-module*

$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' s$	$\lvert\alpha\rvert^2$	b^ϕ
$e_1 + e_2$	$e_1 - e_j,\ 3 \le j \le 6$	$\phi,\ \phi - \alpha$	1	4	$\frac{1}{18}$	$\frac{1}{9}$

p=2

$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i - e_j) : 1 \le i \le 2,\ 3 \le j \le 6; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^2 (-1)^{\nu_i} \pm \sum_3^6 e_i) : 1 \, odd \, \nu_i\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2$
$\mathcal{R}_{\mathfrak{n}^1} = \{\pm(e_1 + e_2);\ \pm(e_i + e_j) :\ 3 \le i < j \le 6; \pm\frac{1}{2}(e_7 - e_8 - e_1 - e_2 + \sum_3^6 (-1)^{\nu_i} e_i) : 1 \, odd \, \nu_i; \pm\frac{1}{2}(e_7 - e_8 + e_1 + e_2 + \sum_3^6 (-1)^{\nu_i} e_i) : 3 \, odd \, \nu_i\}$
$\mathcal{R}_{\mathfrak{n}^2} = \{\pm(e_i + e_j) :\ 4 \le i < j \le 6;\ \pm\frac{1}{2}(e_7 - e_8 \pm \sum_1^3 e_i \mp \sum_4^6 e_i)\}$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' s$	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$e_1 + e_2$	$e_i - e_j,\ i = 1, 2,\ 3 \le j \le 6$	$\phi,\ \phi - \alpha$	1	8	$\frac{1}{18}$	$\frac{2}{9}$
$\mathfrak{n}^2$	$e_1 + e_6$	$e_1 - e_j,\ 3 \le j \le 6$	$\phi,\ \phi - \alpha$	1	4	$\frac{1}{18}$	$\frac{1}{6}$
		$e_2 - e_6$	$\phi,\ \phi + \alpha$		1		
		$\frac{1}{2}(e_7 - e_8 + e_1 - e_2 + \sum_3^6 e_i)$	$\phi,\ \phi - \alpha$		1		
		$\frac{1}{2}(e_7 - e_8 - e_1 + e_2 - \sum_3^6 e_i)$	$\phi,\ \phi + \alpha$		1		

p=3

$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i - e_j) : 1 \leq i \leq 3,\ 4 \leq j \leq 6; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^3 (-1)^{\nu_i} e_i + \sum_4^6 e_i) : 1\, odd\, \nu_i; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^3 (-1)^{\nu_i} e_i - \sum_4^6 e_i) : 2\, odd\, \nu_i' s\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2$
$\mathcal{R}_{\mathfrak{n}^1} = \{\pm(e_i + e_j) :\ 1 \leq i < j \leq 3\, or\, 1 \leq i \leq 3,\ 4 \leq j \leq 6;\quad \pm\frac{1}{2}(e_7 - e_8 + \sum_1^3 (-1)^{\mu_i} e_i + \sum_3^6 (-1)^{\nu_i} e_i) : 2\, odd\, \mu_i' s\, and\, 1\, odd\, \nu_i;$
$\pm\frac{1}{2}(e_7 - e_8 + \sum_1^3 (-1)^{\mu_i} e_i + \sum_3^6 (-1)^{\nu_i} e_i) : 1\, odd\, \mu_i' s\, and\, 2\, odd\, \nu_i\}$
$\mathcal{R}_{\mathfrak{n}^2} = \{\pm(e_i + e_j) :\ 4 \leq i < j \leq 6; \pm\frac{1}{2}(e_7 - e_8 \pm \sum_1^3 e_i \mp \sum_4^6 e_i)\}$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' s$	$\lvert\alpha\rvert^2$	b^ϕ
		$e_i - e_j,\ i = 1, 2,\ j = 4, 5, 6$	$\phi,\ \phi - \alpha$		6		
$\mathfrak{n}^1$	$e_1 + e_2$	$\frac{1}{2}(e_7 - e_8 + e_1 + e_2 - e_3 + \sum_4^6 e_i)$	$\phi,\ \phi - \alpha$	1	1	$\frac{1}{18}$	$\frac{2}{9}$
		$\frac{1}{2}(e_7 - e_8 - e_1 - e_2 + e_3 - \sum_4^6 e_i)$	$\phi,\ \phi + \alpha$		1		
		$e_i - e_j,\ i = 1, 2, 3,\ j = 4, 6$	$\phi,\ \phi - \alpha$		6		
$\mathfrak{n}^2$	$e_4 + e_6$	$\frac{1}{2}(e_7 - e_8 + \sum_1^3 (-1)^{\nu_i} e_i + \sum_4^6 e_i), 1\, odd\, \nu_i$	$\phi,\ \phi - \alpha$	1	3	$\frac{1}{18}$	$\frac{1}{3}$
		$\frac{1}{2}(e_7 - e_8 + \sum_1^3 (-1)^{\nu_i} e_i - \sum_4^6 e_i), 2\, odd\, \nu_i' s$	$\phi,\ \phi + \alpha$		3		

p=4

$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i - e_j) : 1 \leq i \leq 4,\ 5 \leq j \leq 6; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^4 (-1)^{\nu_i} e_i + \sum_5^6 e_i) : 1\, odd\, \nu_i; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^4 (-1)^{\nu_i} e_i - \sum_5^6 e_i) : 3\, odd\, \nu_i' s\}$

$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \mathfrak{n}^3$

$\mathcal{R}_{\mathfrak{n}^1} = \{\pm(e_i + e_j) :\ 1 \leq i < j \leq 4; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^4 (-1)^{\nu_i} e_i \pm e_5 \mp e_6) : 2\, odd\, \nu_i' s\}$
$\mathcal{R}_{\mathfrak{n}^2} = \{\pm(e_5 + e_6)\}$
$\mathcal{R}_{\mathfrak{n}^2} = \{\pm(e_i + e_j) :\ 1 \leq i \leq 4,\ 5 \leq j \leq 6; \pm\frac{1}{2}(e_7 - e_8 - e_6 - e_5 + \sum_1^4 (-1)^{\nu_i} e_i) : 1\, odd\, \nu_i; \pm\frac{1}{2}(e_7 - e_8 + e_6 + e_5 + \sum_1^4 (-1)^{\nu_i} e_i) : 3\, odd\, \nu_i' s\}$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	$No\ of\ \alpha's$	$\lvert\alpha\rvert^2$	b^ϕ
		$e_i - e_j,\ i = 1,2,\ j = 5,6$	$\phi,\ \phi - \alpha$		4		
$\mathfrak{n}^1$	$e_1 + e_2$	$\frac{1}{2}(e_7 - e_8 + e_6 + e_5 \mp e_4 \pm e_3 + e_2 + e_1)$	$\phi,\ \phi - \alpha$	1	2	$\frac{1}{18}$	$\frac{2}{9}$
		$\frac{1}{2}(e_7 - e_8 - e_6 - e_5 \mp e_4 \pm e_3 - e_2 - e_1)$	$\phi,\ \phi + \alpha$		2		
		$e_i - e_j,\ i = 1,2,3,4,\ j = 5,6$	$\phi,\ \phi + \alpha$		8		
$\mathfrak{n}^2$	$e_5 + e_6$	$\frac{1}{2}(e_7 - e_8 + e_6 + e_5 + \sum_1^4 (-1)^{\nu_i} e_i), 1\, odd\, \nu_i$	$\phi,\ \phi - \alpha$	1	4	$\frac{1}{18}$	$\frac{4}{9}$
		$\frac{1}{2}(e_7 - e_8 - e_6 - e_5 + \sum_1^4 (-1)^{\nu_i} e_i), 3\, odd\, \nu_i's$	$\phi,\ \phi + \alpha$		4		
		$e_i - e_6,\ i = 1,2,3,4$	$\phi,\ \phi + \alpha$		1		
		$e_1 - e_5$	$\phi,\ \phi - \alpha$		4		
$\mathfrak{n}^3$	$e_1 + e_6$	$\frac{1}{2}(e_7 - e_8 + e_6 + e_5 + e_1 + \sum_2^4 (-1)^{\nu_i} e_i), 1\, odd\, \nu_i$	$\phi,\ \phi - \alpha$	1	3	$\frac{1}{18}$	$\frac{11}{36}$
		$\frac{1}{2}(e_7 - e_8 - e_6 - e_5 - e_1 + \sum_2^4 (-1)^{\nu_i} e_i), 2\, odd\, \nu_i's$	$\phi,\ \phi + \alpha$		3		

A.9 $\mathfrak{e}_6$

In this Section we consider the bisymmetric triples of the form $(\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathbb{R}, \mathfrak{l})$ and $(\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2, \mathfrak{l})$.
If $e_1, \dots, e_8$ is the canonical basis for $\mathbb{R}^8$, we can write the root system for $\mathfrak{e}_6$ as follows:

$$\mathcal{R} = \{\pm e_i \pm e_j : 1 \leq i < j \leq 5;\ \pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 (-1)^{\nu_i} e_i) : \sum_1^5 \nu_i \text{ is even}\}.$$

Throughout, all the relations for the ν_i's are *mod* 2.
We recall that on $\mathfrak{e}_6$ there is only one root length which is

$$|\alpha|^2 = \frac{1}{12} \tag{A.20}$$

Lemma A.3. *Let* $\phi = \frac{1}{2}(e_8 - e_7 - e_6 \sum_1^5 (-1)^{\nu_i} e_i) \in \mathcal{R}$.
(i)Let $\alpha = \frac{1}{2}(e_8 - e_7 - e_6 \sum_1^5 (-1)^{\mu_i} e_i) \in \mathcal{R}$. *The string* $\phi + n\alpha$ *is either singular or* ϕ, $\phi - \alpha$. *So either* $d_{\alpha\phi} = 0$ *or* 1, *respectively. We have that* $\phi - \alpha$ *is a root if and only if* $\nu_i \neq \mu_i$, *for two indices* $i_1, i_2 \in \{1, \dots, 5\}$ *and in this case,* $\phi - \alpha = (-1)^{\nu_{i_1}} e_{i_1} + (-1)^{\nu_{i_2}} e_{i_2}$.
(ii) Let $\alpha' = (-1)^{\mu_j} e_j + (-1)^{\mu_k} e_k \in \mathcal{R}$, $1 \leq j < k \leq 5$. *The string* $\phi + n\alpha'$ *is either singular or* ϕ, $\phi - \alpha'$. $\phi - \alpha'$ *is a root if and only if* $\alpha' = (-1)^{\nu_j} e_j + (-1)^{\nu_k} e_k$, *for* $1 \leq j < k \leq 5$. *In this case,* $\phi - \alpha' = \frac{1}{2}\big(e_8 - e_7 - e_6 \sum_{\substack{i=1 \\ i \neq j,k}}^5 (-1)^{\nu_i} e_i + (-1)^{\nu_j + 1} e_j + (-1)^{\nu_k + 1} e_k\big)$.[3]

Proof: $\mathcal{R}$ is a subsystem of roots of the root system for $\mathfrak{e}_8$ and thus we use Lemma A.1. For (i), since $\nu_i = \mu_i$, for $i = 6, 7, 8$, the case that $\nu_i = \mu_i$ for precisely two indices in $\{1, \dots, 8\}$ never happens. Hence, $\phi + \alpha$ is never a root. If $\nu_i \neq \mu_i$ for two indices $i_1, i_2 \in \{1, \dots, 5\}$, then $\phi - \alpha$ is a root and $\phi - \alpha = (-1)^{\nu_{i_1}} e_{i_1} + (-1)^{\nu_{i_2}} e_{i_2}$. This case may happen since $\sum_1^5 \nu_i$ and $\sum_1^5 \mu_i$ are even with 5 odd.
(ii) follows directly from (ii) in Lemma A.1.
□

Symmetric Pair A.15. $(\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathbb{R})$.

$\mathfrak{k}$	$\mathcal{R}_\mathfrak{k}$	γ
$\mathfrak{so}_{10}$	$\{\pm e_i \pm e_j : 1 \leq i < j \leq 5\}$	$\frac{2}{3}$

$$\mathcal{R}_\mathfrak{n} = \{\pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 (-1)^{\nu_i} e_i) : \sum_1^5 \nu_i \, is \, even^{(*)}\}$$

(*) there are either 0, 2 *or* 4 *negative signs.*

[3] We may choose $-\alpha$' in which case we obtain $\phi + \alpha'$ instead.

Symmetric Pair A.16. $(\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2)$.

$\mathfrak{k}_i$	$\mathcal{R}_{\mathfrak{k}_i}$	γ_i
$\mathfrak{su}_6$	$\{\pm(e_i - e_j) : 1 \le i < j \le 5;\ \pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 (-1)^{\nu_i} e_i) : 4\, odd\, \nu_i' s\}$	$\frac{1}{2}$
$\mathfrak{su}_2$	$\{\pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 e_i)\}$	$\frac{1}{6}$

$$\mathcal{R}_{\mathfrak{n}} = \{\pm(e_i + e_j) : 1 \le i < j \le 5;\ \pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 (-1)^{\nu_i} e_i) : 2\, odd\, \nu_i' s\}$$

Bisymmetric Triple A.51. $(\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathbb{R}, \mathfrak{u}_5 \oplus \mathbb{R})$. *(Type I)*

$$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i + e_j) : 1 \le i < j \le 5\}$$
$$\mathfrak{n} = \mathfrak{n}^0 \oplus \mathfrak{n}^2 \oplus \mathfrak{n}^4$$
$$\mathcal{R}_{\mathfrak{n}^0} = \{\pm\tfrac{1}{2}(e_8 - e_7 - e_6 + \textstyle\sum_1^5 e_i)\},$$
$$\mathcal{R}_{\mathfrak{n}^2} = \{\pm\tfrac{1}{2}(e_8 - e_7 - e_6 + \textstyle\sum_1^5 (-1)^{\nu_i} e_i) :\ 2\, odd\, \nu_i' s\},$$
$$\mathcal{R}_{\mathfrak{n}^4} = \{\pm\tfrac{1}{2}(e_8 - e_7 - e_6 + \textstyle\sum_1^5 (-1)^{\nu_i} e_i) :\ 4\, odd\, \nu_i' s\}$$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha' s$	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^0$	$\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 e_i)$	*all*	$\phi,\ \phi - \alpha$	1	$\binom{5}{2}$	$\frac{1}{12}$	$\frac{5}{12}$
$\mathfrak{n}^2$	$\frac{1}{2}(e_8 - e_7 - e_6 - e_1 - e_2 + \sum_3^5 e_i)$	$e_1 + e_2$ $e_i + e_j,\ 3 \le i < j \le 5$	$\phi,\ \phi + \alpha$ $\phi,\ \phi - \alpha$	1	1 3	$\frac{1}{12}$	$\frac{1}{6}$
$\mathfrak{n}^4$	$\frac{1}{2}(e_8 - e_7 - e_6 + e_5 - \sum_1^4 e_i)$	$e_i + e_j,\ 1 \le i < j \le 4$	$\phi,\ \phi + \alpha$	1	$\binom{4}{2}$	$\frac{1}{12}$	$\frac{1}{4}$

Bisymmetric Triple A.52. $(\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathbb{R}, \mathfrak{so}_p \oplus \mathfrak{so}_{10-p} \oplus \mathbb{R})$, $p = 2, 4$. *(Type I)*

$$\mathcal{R}_\mathfrak{p} = \{\pm e_i \pm e_j : 1 \le i \le p/2,\ p/2+1 < j < 5\}$$
$$C_\mathfrak{p} \text{ scalar on } \mathfrak{n}$$

$\phi \in \mathcal{R}_\mathfrak{n}$	$\alpha \in \mathcal{R}_\mathfrak{p}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha's$	$\lvert\alpha\rvert^2$	b^ϕ
$\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 (-1)^{\nu_i} e_i)$	$(-1)^{\nu_i} e_i - (-1)^{\nu_j} e_j$, $1 \le i \le p/2,\ p/2+1 \le j \le 5$	$\phi,\ \phi - \alpha$	1	$\frac{p(10-p)}{4}$	$\frac{1}{12}$	$\frac{p(10-p)}{96}$

Bisymmetric Triple A.53. $(\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2, \mathfrak{su}_6 \oplus \mathbb{R})$. *(Type I)*

$$\mathcal{R}_\mathfrak{p} = \{\pm \tfrac{1}{2}(e_8 - e_7 - e_6 + \textstyle\sum_1^5 e_i)\}$$
$$C_\mathfrak{p} \text{ scalar on } \mathfrak{n}$$

$\phi \in \mathcal{R}_\mathfrak{n}$	$\alpha \in \mathcal{R}_\mathfrak{p}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha's$	$\lvert\alpha\rvert^2$	b^ϕ
$\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 + \sum_1^3 e_i)$	$\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 e_i)$	$\phi,\ \phi - \alpha$	1	1	$\frac{1}{12}$	$\frac{1}{24}$

Bisymmetric Triple A.54. $(\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2, \mathfrak{su}_p \oplus \mathfrak{su}_{6-p} \oplus \mathbb{R} \oplus \mathfrak{su}_2)$, $p = 1,2,3$. *(Type I)*

$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i - e_j) : 1 \le i \le 6-p,\ 7-p \le j \le 5; \pm\frac{1}{2}(e_8 - e_7 - e_6 - \sum_{7-p}^{5} e_i + \sum_1^{6-p}(-1)^{\nu_i} e_i) : (5-p)\, odd\, \nu_i's\}$
$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2$, *for* $p = 2,3$ *and* $\mathfrak{n}$ *irreducible* $Ad\,L$*-module for* $p = 1$

	b^ϕ
$\mathfrak{n}^1$	$\frac{p+2}{24}$
$\mathfrak{n}^2$	$\frac{p}{8}$

p=1

$\mathcal{R}_{\mathfrak{l}_1} = \{\pm(e_i - e_j) : 1 \le i < j \le 5\}$
$\mathcal{R}_{\mathfrak{l}_2} = \mathcal{R}_{\mathfrak{k}_2} = \{\pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 e_i)\}$
$\mathcal{R}_{\mathfrak{p}} = \{\pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5(-1)^{\nu_i} e_i) : 4\, odd\, \nu_i's\}$
$\mathfrak{n}$ *irreducible* $Ad\,L$*-module*

$\phi \in \mathcal{R}_{\mathfrak{n}}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha's$	$\lvert\alpha\rvert^2$	b^ϕ
$e_1 + e_2$	$\frac{1}{2}(e_8 - e_7 - e_6 - e_1 - e_2 + \sum_1^3(-1)^{\nu_i} e_i)$, $2\, odd\, \nu_i's$	$\phi,\ \phi - \alpha$	1	3	$\frac{1}{12}$	$\frac{1}{8}$

p=2

$\mathcal{R}_{\mathfrak{l}_1} = \{\pm(e_i - e_j) : 1 \le i < j \le 4;\ \pm\frac{1}{2}(e_8 - e_7 - e_6 + e_5 - \sum_1^4 e_i)\}$
$\mathcal{R}_{\mathfrak{l}_2} = \{\pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 e_i)\}$
$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i - e_5) : 1 \le i \le 4;\ \pm\frac{1}{2}(e_8 - e_7 - e_6 - e_5 + \sum_1^4(-1)^{\nu_i} e_i) : 3\, odd\, \nu_i's\}$
$\mathcal{R}_{\mathfrak{n}^1} = \{\pm(e_i + e_j) : 1 \le i < j \le 4; \pm\frac{1}{2}(e_8 - e_7 - e_6 + e_5 + \sum_1^4(-1)^{\nu_i} e_i) : 2\, odd\, \nu_i's\}$
$\mathcal{R}_{\mathfrak{n}^2} = \{\pm(e_i + e_5) : 1 \le i \le 4; \pm\frac{1}{2}(e_8 - e_7 - e_6 - e_5 + \sum_1^4(-1)^{\nu_i} e_i) : 1\, odd\, \nu_i\}$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	*No of* $\alpha's$	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$e_1 + e_2$	$e_i - e_5$, $i = 1,2$ $\frac{1}{2}(e_8 - e_7 - e_6 - e_5 \pm e_3 \mp e_4 - e_1 - e_2)$	$\phi,\ \phi - \alpha$ $\phi,\ \phi + \alpha$	1	2 2	$\frac{1}{12}$	$\frac{1}{6}$
$\mathfrak{n}^2$	$e_1 + e_5$	$e_i - e_5$, $i = 2,3,4$ $\frac{1}{2}(e_8 - e_7 - e_6 - e_5 + \sum_2^4(-1)^{\nu_i} e_i - e_1)$, $2\, odd\, \nu_i's$	$\phi,\ \phi + \alpha$	1	3 3	$\frac{1}{12}$	$\frac{1}{4}$

p=3

$\mathcal{R}_{\mathfrak{l}_1} = \{\pm(e_i - e_j) : 1 \le i < j \le 3;\ \pm(e_4 - e_5); \pm\frac{1}{2}(e_8 - e_7 - e_6 \pm e_5 \mp e_4 - \sum_1^3 e_i)\}$
$\mathcal{R}_{\mathfrak{l}_2} = \{\pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 e_i)\}$
$\mathcal{R}_{\mathfrak{p}} = \{\pm(e_i - e_j) : i = 1,2,3,\ j = 4,5;\ \pm\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 + \sum_1^3 (-1)^{\nu_i} e_i) :\ exactly\, 2\, odd\, \nu_i' s\}$
$\mathcal{R}_{\mathfrak{n}^1} = \{\pm(e_i + e_j) : 1 \le i < j \le 3\, or\ 1 \le i \le 3,\ j = 4,5;\quad \pm\frac{1}{2}(e_8 - e_7 - e_6 + e_5 + e_4 + \sum_1^3 (-1)^{\nu_i} e_i) : 2\, odd\, \nu_i' s;$
$\pm\frac{1}{2}(e_8 - e_7 - e_6 \pm e_5 \mp e_4 + \sum_1^3 (-1)^{\nu_i} e_i) : 1\, odd\, \nu_i\}$
$\mathcal{R}_{\mathfrak{n}^2} = \{\pm(e_4 + e_5);\ \pm\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 + \sum_1^3 e_i)\}$

$\mathfrak{n}^i$	$\phi \in \mathcal{R}_{\mathfrak{n}^i}$	$\alpha \in \mathcal{R}_{\mathfrak{p}}^+$	$\phi + n\alpha$	$d_{\alpha\phi}$	$No\ of\ \alpha' s$	$\lvert\alpha\rvert^2$	b^ϕ
$\mathfrak{n}^1$	$e_1 + e_2$	$e_i - e_j,\ i = 1,2, j = 4,5$	$\phi,\ \phi - \alpha$	1	4	$\frac{1}{12}$	$\frac{5}{24}$
		$\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 + e_3 - e_2 - e_1)$	$\phi,\ \phi + \alpha$		1		
$\mathfrak{n}^2$	$e_4 + e_5$	$e_i - e_j,\ i = 1,2,3, j = 4,5$	$\phi,\ \phi + \alpha$	1	6	$\frac{1}{12}$	$\frac{9}{24}$
		$\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 + \sum_1^3 (-1)^{\nu_i} e_i),\ 2\, odd\, \nu_i' s$			3		

Bisymmetric Triple A.55. $(\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2, \mathfrak{su}_p \oplus \mathfrak{su}_{6-p} \oplus \mathbb{R} \oplus \mathbb{R})$, $p = 1, 2, 3$. *(Type II)*

$$\begin{aligned}
\mathcal{R}_{\mathfrak{p}} = \;& \{\pm(e_i - e_j) : 1 \leq i \leq 6-p,\, 7-p \leq j \leq 5; \\
& \pm\tfrac{1}{2}(e_8 - e_7 - e_6 - \textstyle\sum_{7-p}^{5} e_i + \sum_1^{6-p} (-1)^{\nu_i} e_i) : (5-p)\, odd\, \nu_i' s\}
\end{aligned}$$

$\mathcal{R}_{\mathfrak{p}_2} = \{\pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 e_i)\}$

decomposition of $\mathfrak{n}$ *as in* $A.54$

$\mathfrak{n}^i$	b_1^ϕ	b_2^ϕ
$\mathfrak{n}^1$	$\frac{p+2}{24}$	$\frac{1}{24}$
$\mathfrak{n}^2$	$\frac{p}{8}$	$\frac{1}{24}$

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Notation List

Common notation:

A_g Conjugation by $g \in G$ in a Lie group G

Ad_g Adjoint action of $g \in G$ in the Lie algebra $\mathfrak{g}$ of a Lie group G

ad_X Adjoint linear map of $X \in \mathfrak{g}$

$Kill$ the Killing form of a Lie algebra $\mathfrak{g}$

$B = -Kill$

C_V the Casimir operator of a subspace V of a Lie algebra $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$

$c_{V,U}$ the eigenvalue of a Casimir operator C_V on a subspace U

R the curvature of a Riemannian metric

Ric the Ricci curvature of a Riemannian metric

K the sectional curvature of a Riemannian metric

$h^*(\mathfrak{g})$ the dual Coxeter number of a Lie algebra $\mathfrak{g}$

$\mathfrak{R}$ the system of roots of a Lie algebra $\mathfrak{g}$

$\mathfrak{R}_K$ the system of restricted roots to a subgroup of maximal rank K

S^+ the subset of positive roots of a set of roots S

$\mathfrak{g}^{\mathbb{C}}$ the complexification of a Lie algebra $\mathfrak{g}$

$V^{\mathbb{C}}$ the complexification of a vector space V

id_V the identity map of a vector space V

0_V the null map of a vector space V

Notation for homogeneous fibrations:

G compact connected semisimple Lie group

K, L compact closed non-trivial subgroups of G such that $L \subsetneq K \subsetneq G$

$\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{l}$ the Lie algebras of G, K and L

$M = G/L$

$N = G/K$

$F = K/L$

g_M an adapted metric metric on M with respect to a fibration $F \to M \to N$

g_N the projection of an adapted metric g_M onto the base space N as above

g_F the restriction of an adapted metric g_M to the fiber space F as above

$\mathfrak{n}$ $Ad\,K$-invariant complement of $\mathfrak{k}$ on $\mathfrak{g}$

$\mathfrak{p}$ $Ad\,L$-invariant complement of $\mathfrak{l}$ on $\mathfrak{k}$

$\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{n}$ $Ad\,L$-invariant complement of $\mathfrak{l}$ on $\mathfrak{g}$

$\mathfrak{n} = \mathfrak{n}_1 \oplus \dots \mathfrak{n}_n$ decomposition of $\mathfrak{n}$ into pairwise inequivalent irreducible $Ad\,K$-submodules

$\mathfrak{n} = \mathfrak{n}^1 \oplus \dots \mathfrak{n}^{n'}$ decomposition of $\mathfrak{n}$ into pairwise inequivalent irreducible $Ad\,L$-submodules

$\mathfrak{p} = \mathfrak{p}_1 \oplus \dots \mathfrak{p}_s$ decomposition of $\mathfrak{p}$ into pairwise inequivalent irreducible $Ad\,L$-submodules

$C_\mathfrak{g}$, $C_\mathfrak{k}$, $C_\mathfrak{l}$ the Casimir operator of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{l}$, respectively

$C_{\mathfrak{p}_a}$ the Casimir operator of $\mathfrak{p}_a$, $a = 1, \dots, s$

$C_{\mathfrak{n}_i}$ the Casimir operator of $\mathfrak{n}_i$, $i = 1, \dots, n$

$c_{\mathfrak{l},a}$ the eigenvalue of $C_\mathfrak{l}$ on $\mathfrak{p}_a$, $a = 1, \dots, s$

$c_{\mathfrak{l},\mathfrak{p}}$ the eigenvalue of $C_\mathfrak{l}$ on $\mathfrak{p}$, when $\mathfrak{p}$ is $Ad\,L$-irreducible

$c_{\mathfrak{k},i}$ the eigenvalue of $C_\mathfrak{k}$ on $\mathfrak{n}_i$, $i = 1, \dots, n$

$c_{\mathfrak{k},\mathfrak{n}}$ the eigenvalue of $C_\mathfrak{k}$ on $\mathfrak{n}$, when $\mathfrak{n}$ is $Ad\,K$-irreducible

$c_{\mathfrak{n}_i,a}$ the constant defined by $Kill(C_{\mathfrak{n}_i}\cdot,\cdot)\,|_{\mathfrak{p}_a\times\mathfrak{p}_a} = c_{\mathfrak{n}_i,a} Kill\,|_{\mathfrak{p}_a\times\mathfrak{p}_a}$, $a = 1, \dots, s$

γ_a the constant defined by $Kill_\mathfrak{k}\,|_{\mathfrak{p}_a\times\mathfrak{p}_a} = \gamma_a Kill\,|_{\mathfrak{p}_a\times\mathfrak{p}_a}$

γ the constant defined by $Kill_\mathfrak{k}\,|_{\mathfrak{p}\times\mathfrak{p}} = \gamma Kill\,|_{\mathfrak{p}\times\mathfrak{p}}$, when $\mathfrak{p}$ is $Ad\,L$-irreducible

b_a^i the eigenvalue of $C_{\mathfrak{p}_a}$ on $\mathfrak{n}_i$, $i = 1, \dots, n$, when this eigenvalue exists (the indices are dropped in the case of irreducibility as above)

b_a^ϕ the constant defined by $b_a^\phi = B(C_{\mathfrak{p}_a} X_\phi, X_{-\phi}) = Kill(C_{\mathfrak{p}_a^\mathbb{C}} E_\phi, E_{-\phi})$ for a root ϕ; for $\phi \in \mathcal{R}_\mathfrak{n}$, it represents an eigenvalue of $C_{\mathfrak{p}_a}$ on $\mathfrak{n}$